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String theory and perturbative analysis of strings in AdS space

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Resumo

A teoria de cordas é uma candidata muito boa para ser a teoria da gravidade quântica. A baixas energias podemos encontrar os constituintes básicos do nosso universo: relatividade geral, campos escalares, teorias de gauge e fermiões quirais. Por outro lado, modernamente, a teoria de cordas é também vista como uma ferramenta para teoria quântica de campo avançada pois percorre temas como a teoria de campo conforme, simetrias de gauge até ao surpreendente resultado da dualidade AdS/CFT, atribuindo naturalmente um caracter interdisciplinar à teoria.

Neste trabalho, motivamos e introduzimos a teoria classicamente para ser posteriormente quantificada de uma forma covariante e não-covariante. Estudamos também teoria de campo conforme apresentado diretamente as relações com teoria de cordas, exibindo as suas técnicas e mostrando as suas particularidades quando comparadas com outras teorias de campo. Métodos de integrais de caminhos e quantificação BRST são também usados como uma abordagem alternativa à teoria. Estudamos também cordas em *backgrounds* não triviais onde obtemos as equações de Einstein no vácuo e apresentamos a primeira correção quântica obtida usando teoria de cordas, impondo invariância de Weyl.

Um exemplo da dualidade AdS/CFT é a equivalência exata entre teoria de cordas do tipo IIB compactificada em $AdS_5 \times S^5$ e a teoria $\mathcal{N} = 4$ Yang-Mills supersimétrico quatro dimensional. Em AdS_5 a dimensão dos operadores vertex muda devido à curvatura. É feito um estudo perturbativo considerando os primeiros termos das expanções no integral de caminho e nas soluções da equação de Klein Gordon em AdS_5. Como exemplo concreto calculamos a função de 2-pontos do Lagrangeano explicitamente e apresentamos a primeira correção à sua dimensão.

Abstract

String theory is a very good candidate to be the theory of quantum gravity. At low-energies one can find the basic constituents of our universe: general relativity, scalar fields, gauge theories and chiral fermions. On the other hand, modernly, string theory is also regarded as an advanced quantum field theory tool ranging from conformal field theory, gauge symmetry until the remarkable result of AdS/CFT duality, giving naturally an interdisciplinary aspect to the theory.

In this framework, we motivate and introduce the theory classically to be quantized in a covariant and non-covariant manner. We also study conformal field theory presenting direct connections with string theory, exhibiting its techniques and showing its particularities compared with other field theories. Path integral methods and BRST quantization are also used to get another approach to the theory. We also study strings on (non trivial) background fields obtaining Einstein equations in vacuum and presenting the first quantum correction arising from string theory by imposing Weyl invariance.

An example of AdS/CFT duality is the exact equivalence of type IIB string theory compactified in $AdS_5 \times S^5$ with four dimensional $\mathcal{N} = 4$ supersymetric Yang-Mills theory. In AdS_5 the dimension of the vertex operators changes due to curvature. A perturbative study of this is made by picking the first terms of the expansions in the path integral and in the Klein Gordon equation solutions in AdS_5. As a concrete example, the 2-point function of the Lagrangian is calculated explicitly and the first correction to its dimension is presented.

Contents

1. Introduction

Historically, string theory has its roots in the attempt of constructing a theory to describe hadrons, long before the interpretation of "strings" was adopted. In 1968 Veneziano publishes a simple analytic formula that exhibits duality with linear Regge trajectories[1]. This generated a wide range of results with successive generalizations. In 1971 Lovelace proposed for the first time the famous result of the 26 dimensions[2]. Only later the idea of closed and open strings arrived.

Superstring theory appeared in the same year of 1971 when Ramond constructed what we could call the Dirac equation for strings[4]. Nowadays one of the hottest topics in string theory is AdS/CFT duality that relates theories with gravity in a certain space (AdS) with a conformal field theory defined on its boundaries. This result was first conjectured by Maldacena in 1997[5].

This work does not intend to follow the history of the theory but rather to present a detailed and self-contained introduction to string theory. Taking into account the AdS/CFT duality, this study intends to clarify results and techniques of string theory in order to understand the spectrum of the string in AdS space, which is something not fully understood yet.

In chapter 2 we start by generalizing what would be an action of a n dimensional object. Motivated by the symmetry properties of the case $n = 1$, strings, we move on to its quantization where it is possible to obtain for the first time the critical dimension of 26 (the dimension for which string theory can exist in a consistent way). The string is quantized in two ways. The two different methods deal in a different way with the appearance of states with negative norm (ghosts), giving two different and equivalent ways to look at the theory. Particularly we find as elements of the spectrum a tachyon (negative squared mass state) and the graviton (plus two other massless states).

In chapter 3 we explore several features of conformal field theory with particular emphasis on string theory. We explore the important tools of Operator Product Expansion, Noether theorem and the Ward identities. We also study a special kind of operators, quasi-primary and primary operators, defined by the way they transform under a conformal transformation, and we see that 2- and 3-point functions are completely fixed for them. Conformal field theories also have a wide range of symmetry explicit in the Virasoro Algebra. These algebra suffers from an anomaly, Weyl anomaly, that constitutes a problem to be solved in following chapters. The chapter ends with an important as well as surprising result: the state-operator map. This map relates states of the theory with operators, called vertex operators, objects of crucial importance in string theory. We will see how can we take this map explicitly.

In chapter 4 the spectrum of the string is obtained in a more modern way using path integral techniques and the method of BRST quantization. Even though this is not explicitly present, this method gives the same result as the other two.

In all the previous chapters we always consider to be in flat space. In chapter 5 we introduce a general metric and try to understand its meaning. We see that this corresponds to the insertion of a coherent state of gravitons and by requiring Weyl invariance we obtain Einstein equations of General Relativity in vacuum. We introduce the other two fields that are in an equal footing with the graviton in the action obtaining the respective equations. This procedure also allow us to obtain corrections to Einstein equations in vacuum.

In chapter 6 we study how to calculate the correction to the dimension of vertex operators caused by the curvature ofAdS_5. We find the correct coordinates to perform this task: flat metric up to second order. With this we see how to obtain mean values (perturbatively) in AdS_5 through calculations in Minkowski space only. To obtain 2-point functions to compare with the flat space case, it is also necessary to take into account how the operators change. This is done by expanding the solutions of the Klein Gordon equation. As an example of this procedure the 2-point function of the Lagrangian in AdS_5 is calculated explicitly predicting the anomalous dimension of this operator.

2. Free bosonic strings

2.1. Classical String

The action of a relativistic point particle is well known and it is proportional to the length of the world line:

$$S = -m \int ds \tag{2.1.1}$$

We may try to generalize this action for n-dimensional objects, with, of course, $n+1 \leq D$ that is the dimension of the space. Such an object would sweet out a $n+1$-dimensional surface, giving the obvious generalization for the action. This surface will have an induced metric. Let σ^a be its parametrization $(a = 1, ..., n+1)$, then:

$$ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta = g_{\alpha\beta} \frac{\partial X^\alpha}{\partial \sigma^a} \frac{\partial X^\beta}{\partial \sigma^b} d\sigma^a d\sigma^b = \gamma_{ab} d\sigma^a d\sigma^b \tag{2.1.2}$$

$X^\mu(\sigma)$ constitutes a map between the $n+1$ manifold and the physical space-time.γ_{ab} is the induced metric on the surface.

An action that is proportional to the area of the object is simply:

$$S = -T \int d^{n+1}\sigma \sqrt{\gamma} \tag{2.1.3}$$

The problem with this action is the square root that turns path integral quantization very difficult. We can avoid this problem by introduction of another field: the metric h^{ab}

$$S = -\frac{T}{2} \int d^{n+1}\sigma \sqrt{h} \left(h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - (n-1) \right) \tag{2.1.4}$$

The equation for the metric is given by:

$$-\frac{2}{T} \frac{\delta S}{\delta h^{ab}} = -\frac{1}{2} \sqrt{h} h_{ab} \left(h^{cd} \partial_c X^\mu \partial_d X^\nu g_{\mu\nu} - (n-1) \right) + \sqrt{h} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} = 0 \tag{2.1.5}$$

The above equation is solved by $h_{ab} = \partial_a X^\mu \partial_b X_\mu$ and we recover 2.1.3. This action has an important property: it is parametrization invariant, i.e., does not depend on the particular choice of coordinates to describe the manifold. This gauge invariance has $(n+1)$ independent reparametrizations that can be used to gauge away $(n+1)$independent components of the total of $\frac{1}{2}(n+1)(n+2)$ independent components of the tensor h^{ab}, giving $\frac{1}{2}n(n+1)$ components left. For $n > 0$ this does not eliminate the h dependence. But now, consider a Weyl scaling of the metric:

$$h_{ab} \to \Lambda(\sigma)\, h_{ab} \tag{2.1.6}$$

This will also imply $h^{ab} \to \Lambda^{-1} h^{ab}$ and $\sqrt{h} \to \Lambda^{\frac{1}{2}(n+1)}$.

Precisely for $n = 1$ (strings) the action 2.1.4 loses the extra term $n-1$, and we have Weyl invariance. Furthermore it gives an extra symmetry we can use to fix components of the metric. In this case $\frac{1}{2} n(n+1) = 1$ and we can eliminate the independent component left from reparametrization. The Polyakov action is the above action in Minckowski space:

$$S = -\frac{T}{2} \int d^{n+1}\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{2.1.7}$$

These are local symmetries. Explicitly their infinitesimal transformation are:

- Reparametrization invariance:

$$\begin{gathered} \delta\sigma^a = -\xi^a(\sigma) \\ \delta X^\mu = \xi^a \partial_a X^\mu \\ \delta h^{ab} = \xi^c \partial_c h^{ab} - \partial_c \xi^a h^{cb} - \partial_c \xi^b h^{ac} \\ \delta\sqrt{h} = \partial_a\left(\xi^a \sqrt{h}\right) \end{gathered} \tag{2.1.8}$$

- Weyl scaling:

$$\delta h^{ab} = \varepsilon(\sigma)\, h^{ab}$$

We have also one global symmetry on the X fields:

- Poincaré invariance:

$$\delta X^\mu = a^\mu{}_\nu X^\mu + b^\mu \tag{2.1.9}$$

The local symmetries will prove to be crucial in string theory. In fact they are not really physical symmetries, they are just equivalent ways of describing the system. In the next section we shall fix the gauge.

2.1.1. Gauge fixing and modes expansions

The symmetries of the Polyakov action are enough to fix h_{ab} as a flat metric. We now write the factor T, called the string tension, in another standard way.

$$T = \frac{1}{2\pi\alpha'} \tag{2.1.10}$$

In flat space the Polyakov action simplifies to

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_a X^\mu \partial^a X_\mu \tag{2.1.11}$$

turning the field equation for X^μ simply:

$$\partial_a \partial^a X^\mu = 0 \tag{2.1.12}$$

There is also a surface term, when varying the action, that we will leave for later use

$$\int d\tau \left[\frac{\partial X^\mu}{\partial \sigma}\delta X^\mu\right]_{\sigma=0}^{\sigma=\pi} = 0 \tag{2.1.13}$$

The choice $\sigma = [0,\pi]$ it is just a matter of convenience.

Despite of the gauge fixing the equation for the metric h_{ab} must also be fulfilled. This means that the energy momentum tensor must vanish.

$$T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{ab}} = 0 \tag{2.1.14}$$

The variation of the action was calculated in 2.1.5. Particularizing for the Minckowski metric in two dimensions:

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\eta_{ab}\partial^c X^\mu \partial_c X_\mu \tag{2.1.15}$$

This, along with the wave equation 2.1.12, constitutes the set of equations that we need to solve the problem classically. Using $\sigma^0 = \tau$, $\sigma^1 = \sigma$, primes for σ derivatives and dots for τ derivatives they are:

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0 \tag{2.1.16}$$

$$\begin{gathered} X' \cdot \dot{X} = 0 \\ \tfrac{1}{2}\left(X'^2 + \dot{X}^2\right) = 0 \end{gathered} \tag{2.1.17}$$

The last two equations are constraints for the first wave equation. The wave equation can be easily solved with a change of variables, the lightcone coordinates:

$$\sigma^\pm = \tau \pm \sigma \tag{2.1.18}$$

The 2 dimensional metric has only off diagonal elements: $\eta_{+-} = \eta_{+-} = -\frac{1}{2}$. The equations of motions and the constraints are:

$$\begin{gathered} \partial_+ \partial_- X^\mu = 0 \\ (\partial_+ X)^2 = (\partial_- X)^2 = 0 \end{gathered} \tag{2.1.19}$$

The solutions is simply:

$$X^\mu(\sigma,\tau) = X_L^\mu\left(\sigma^+\right) + X_R^\mu(\sigma^-) \tag{2.1.20}$$

where we call respectively the left and right modes. Now we have to distinguish between the cases of closed and open strings in the boundary conditions.

Closed Strings For closed strings, periodicity in the fields must be required:

$$X^\mu(\sigma,\tau+\pi) = X^\mu(\sigma,\tau) \tag{2.1.21}$$

This automatically fulfills the condition 2.1.13.

Periodicity on the field does not mean that we should have periodicity in the modes, but it is easy to see that they must be periodic up to a constant (that must be the same for right and left modes). The general solution is:

$$\begin{aligned} X_L^\mu(\sigma^+) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^+ + i\left(\tfrac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_n \tfrac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} \\ X_R^\mu(\sigma^-) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^- + i\left(\tfrac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_n \tfrac{1}{n}\alpha_n^\mu e^{-2in\sigma^-} \end{aligned} \tag{2.1.22}$$

Particular normalization factors, as $\frac{1}{n}$ for example , were used. For reality of the field X^μ we should add the condition:

$$\tilde{\alpha}_n^\mu = \left(\tilde{\alpha}_{-n}^\mu\right)^*,\ \alpha_n^\mu = \left(\alpha_{-n}^\mu\right)^* \tag{2.1.23}$$

The quantity x^μ can be interpreted as the center of mass while p^μ can be seen as the momentum. The α's are the modes of oscillation.

We can also calculate the equal times Poisson brackets that will be useful later

$$\begin{aligned} \left[X^\mu(\sigma), X^\nu(\sigma')\right]_{PB} &= \left[\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\right]_{PB} = 0 \\ \left[\dot{X}^\mu(\sigma), X^\mu(\sigma')\right]_{PB} &= \tfrac{1}{T}\delta(\sigma-\sigma')\,\eta^{\mu\nu} \end{aligned} \tag{2.1.24}$$

The non zero commutators among the expansion parameters are:

$$\begin{aligned} \left[x^\mu, p^\mu\right]_{PB} &= -\eta^{\mu\nu} \\ \left[\alpha_m^\mu, \alpha_n^\nu\right] = \left[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\right] &= im\delta_{m+n}\eta^{\mu\nu} \end{aligned} \tag{2.1.25}$$

Open strings The boundary condition for open strings is given by condition 2.1.13 forbidding the momentum to flow off through the ends:

$$\frac{\partial X^\mu}{\partial\sigma}(\pi,\tau) = \frac{\partial X^\mu}{\partial\sigma}(0,\tau) = 0 \tag{2.1.26}$$

To solve the wave equation with this boundary conditions one can extend the domain of X^μ by letting σ take values from 0 to 2π and defining its values for $\sigma \in [\pi, 2\pi]$ by $X^\mu(\sigma,\tau) = X^\mu(2\pi - \sigma, \tau)$. With this extension X^μ is periodic with periodicity 2π. The condition that the normal derivative vanishes at π guarantees continuity of the derivative of the extension. We can use the same procedure as above to obtain the mode expansion of the open string solution.

$$
\begin{aligned}
X^\mu_{L/R}(\sigma^\pm) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^\pm + i\left(\tfrac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_n \tfrac{1}{n}\alpha^\mu_n e^{-in\sigma^\pm} \\
\Rightarrow X^\mu(\sigma,\tau) &= x^\mu + \alpha' p^\mu \tau + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \tfrac{1}{n}\alpha^\mu_n e^{-in\tau}\cos(n\sigma)
\end{aligned}
\tag{2.1.27}
$$

The reality condition 2.1.23 still applies.

In general we can condensate in one expression the two cases:

$$
\left.
\begin{aligned}
X^\mu_L(\sigma^+) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^+ + i\left(\tfrac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n\neq 0} \tfrac{1}{n}\ \tilde{\alpha}^\mu_n\ e^{-ins\sigma^+} \\
X^\mu_R(\sigma^-) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^- + i\left(\tfrac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n\neq 0} \tfrac{1}{n}\ \alpha^\mu_n\ e^{-ins\sigma^-}
\end{aligned}
\right\{
\begin{array}{c}
\text{Closed Strings}: s = 2 \\
\\
\text{Open Strings}: s = 1,\ \tilde{\alpha}^\mu_n = \alpha^\mu_n
\end{array}
\tag{2.1.28}
$$

From now we may refer to the zero modes α^μ_0 and $\tilde{\alpha}^\mu_0$ .They are defined in a way that they can be absorbed on the expansion of the derivatives[1].

$$
\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \frac{1}{s}\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} p^\mu \tag{2.1.29}
$$

The energy momentum tensor tensor in 2.1.15 must take the form of an expansion too. In these coordinates we have only two non-zero components

$$
T_{++} = s^2\alpha' \sum_m \tilde{L}_m e^{-ism\sigma^+},\ T_{--} = s^2\alpha' \sum_m L_m e^{-ism\sigma^-} \tag{2.1.30}
$$

It is therefore a traceless tensor ($T_{ab}\eta^{ab} = 2T_{+-}\eta^{+-} = 0$). Inserting this on constraints in 2.1.19 the conditions read

$$
\sum_m L_m e^{-ism\sigma^-} = \sum_m \tilde{L}_m e^{-isn\sigma^+} = 0 \Rightarrow L_m = \tilde{L}_m = 0 \tag{2.1.31}
$$

We can get an expression for this Fourier components substituting the expansion for X

$$
\begin{aligned}
L_n &= \tfrac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m \\
\tilde{L}_n &= \tfrac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m
\end{aligned}
\tag{2.1.32}
$$

[1] For example, with this definition we can write $\frac{\partial X^\mu_L}{\partial\sigma^+}(\sigma^+) = s\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}^\mu_n e^{-sin\sigma^+}$

The Poisson brackets between them give the Virasoro algebra:

$$[L_m, L_n]_{PB} = i\,(m-n)\,L_{m+n} \tag{2.1.33}$$

The elements L_m are called the Virasoro generators, with analogous expression for $\tilde{L}_m$. There is a special case, the condition $L_0 = \tilde{L}_0 = 0$ contains the square of the momentum which gives the mass-shell condition ($p^\mu p_\mu = -M^2$)

$$M^2 = s^2 \frac{1}{\alpha'} \sum_{m>0} \alpha_m \cdot \alpha_{-m} = s^2 \frac{1}{\alpha'} \sum_{m>0} \tilde{\alpha}_m \cdot \tilde{\alpha}_{-m} \tag{2.1.34}$$

2.1.2. Symmetries of the Polyakov Action

The action 2.1.11 has Poincaré invariance. We can find associated conserved currents in the usual way considering the transformation

$$X^\mu(\sigma) \to X^\mu(\sigma) + \epsilon(\sigma)\,\delta X^\mu(\sigma) \tag{2.1.35}$$

that is a symmetry for constant ϵ. Then, the variation of the action is

$$\delta S = \int d^2\sigma J^a \partial_a \epsilon \tag{2.1.36}$$

with J^a being the conserved current. One can then obtain:

- Translations: $P^a_\mu = T\partial^a X_\mu$
- Lorentz transformations: $J^a_{\mu\nu} = T\,(X_\mu \partial^a X_\nu - X_\mu \partial^a X_\nu)$

Conserved currents give rise to conserved charges. The angular momentum is:

$$M_{\mu\nu} = \int d\sigma \left(X_\mu \frac{\partial X_\nu}{\partial \tau} - X_\nu \frac{\partial X_\mu}{\partial \tau} \right) = l_{\mu\nu} + S_{\mu\nu} + \tilde{S}_{\mu\nu} \tag{2.1.37}$$

This expression explicits the two origins of the total angular momentum: the angular momentum of the string plus the angular momentum due to excited oscillator modes

$$\begin{gathered} l_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \\ S^{\mu\nu} = -i \sum_{n>0} \tfrac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right) \end{gathered} \tag{2.1.38}$$

After an overview of the classical results we will now quantize the theory.

2.2. The Quantum String

In some of the expressions above we already omitted the correspondent equation with the ˜ over the operators. From now on, unless it is explicitly useful, we will always omit the analogue counterpart with ˜.

In this section the classical theory is quantized. We will start by following a procedure that is not explicitly Lorentz invariant named "Light-cone quantization". Even though the theory is Lorentz invariant classically, if this property is not guaranteed during quantization, it may not survive. The cost is to force Lorentz invariance after the quantization. Nevertheless it will avoid the appearance of ghosts: states with negative norm. In other words the resultant Fock space is not positive definite demanding extra conditions to find the subspace of the so called "Physical States". Other possible procedure is known as "Old covariant quantization". In this case we find ghosts and we do not have a clear way to treat them. Nonetheless a positive definite space is obtained by choice of suitable conditions. The two procedures are equivalent.

The modern way to quantize strings is the path integral quantization presented on chapter 4. Again the three ways to quantize the theory prove to generate the same space.

2.2.1. Light-Cone quantization

By picking the Minckowski metric it seems that all gauge freedom was used. Actually a gauge freedom still exists. T transformation

$$\sigma^+ \to \tilde{\sigma}^+\left(\sigma^+\right),\ \sigma^- \to \tilde{\sigma}^-\left(\sigma^-\right) \tag{2.2.1}$$

changes the metric by a multiplicative function

$$\eta_{ab} \to \Omega\left(\sigma\right)\eta_{ab} \tag{2.2.2}$$

that can be eliminated by Weyl invariance. Previously we said that the gauge freedom was exactly right (2 reparametrization invariance plus 1 Weyl invariance) to cancel the three independent components of the metric. The reason for an apparent extra freedom to exist lies on the nature of this transformations. In fact they are just one-variable transformations giving a zero measure in the set of gauge symmetries. In the original variables the transformations are explicitly

$$\begin{aligned} \tilde{\tau} &= \tfrac{1}{2}\left(\tilde{\sigma}^+\left(\tau+\sigma\right)+\tilde{\sigma}^-\left(\tau-\sigma\right)\right) \\ \tilde{\sigma} &= \tfrac{1}{2}\left(\tilde{\sigma}^+\left(\tau+\sigma\right)-\tilde{\sigma}^-\left(\tau-\sigma\right)\right) \end{aligned} \tag{2.2.3}$$

meaning that we can view $\tilde{\tau}$ as a solution of the free massless equation 2.1.16. Adopting a particular solution of the equation fixes the gauge.

We are going to adopt the light-cone gauge. The light-cone coordinates are defined as:

$$X^{\pm} = \sqrt{\frac{1}{2}}\left(X^0 \pm X^{D-1}\right) \tag{2.2.4}$$

We just broke the Lorentz invariant aspect of the theory. The rest of the coordinates will be denoted by X^i with $i = 1, ..., D-2$. The metric is:

$$ds^2 = -2dX^+dX^- + \sum_i dX^i dX^i \tag{2.2.5}$$

Because they obey the same equation, we can take $\tilde{\tau}$ to be proportional to X^+ plus a constant. The light-cone gauge corresponds to write

$$X^+ = x^+ + \alpha' p^+ \tau \tag{2.2.6}$$

But now this fixes X^-too in terms of other fields. From the first equation of 2.1.19 we see that

$$X^- = X_L^-\left(\sigma^+\right) + X_R^-\left(\sigma^-\right) \tag{2.2.7}$$

from the following two comes

$$\begin{aligned} \partial_+ X_L^- &= \tfrac{1}{\alpha' p^+} \sum_i \partial_+ X^i \partial_+ X^i \\ \partial_- X_R^- &= \tfrac{1}{\alpha' p^+} \sum_i \partial_- X^i \partial_- X^i \end{aligned} \tag{2.2.8}$$

Using the usual modes expansion 2.1.28 for X^-we fix all the constants but x^-

$$\alpha_n^- = \left(\frac{1}{2\alpha'}\right)^{\frac{1}{2}} \frac{1}{p^+} \sum_m \sum_i \alpha_{n-m}^i \alpha_m^i \tag{2.2.9}$$

From here the mass condition can be written with just the "i-modes" that are called the transverse oscillators.

$$M^2 = \frac{s^2}{\alpha'} \sum_{m>0} \sum_i \alpha_{-m}^i \alpha_m^i \tag{2.2.10}$$

We took care of the residual symmetry that was the potential source of ghosts. We can now pass to quantization.

2.2.1.1. Quantization

The degrees of freedom left are x^-, p^+, x^i, p^i, α_m^i , $\tilde{\alpha}_m^i$. The x^+ quantity is not a degree of freedom since it can be absorbed by a shift in τ. Knowing their Poisson Brackets from 2.1.25 we can write their commutation relation by a standard way of passing from classical to quantum theory, making the replacement $[...]_{P.B.} \to -i\,[...]$:

$$\begin{aligned} \left[x^-, p^+\right] &= -i, \ \left[x^i, p^j\right] = -i\delta^{ij} \\ \left[\alpha_m^i, \alpha_n^j\right] &= \left[\tilde{\alpha}_m^i, \tilde{\alpha}_n^j\right] = m\delta_{m+n}\delta^{ij} \end{aligned} \tag{2.2.11}$$

from which it also follows $\left[x^+, p^-\right] = -i$

The Hilbert space is constructed with momentum p^μ and creation and annihilation operators (α^i_{-m} and α^i_m with $m > 0$ respectively) just for the transverse modes. This space is positive definite and thus, it has no ghosts.

The method just used to quantize the classical oscillators, $[...]_{P.B.} \rightarrow -i\,[...]$, may be applied to other physical quantities, but one should be careful with ordering issues. Now the physical quantities do not always commute and ambiguities may appear in the quantum expressions. Namely let us recall the Virasoro algebra in equation 2.1.33. There are no order ambiguities in the expations of the L_m (equations 2.1.32) except for the case of $m = 0$. We always put it normal ordered but we have to introduce a constant a^X on the expression to be determined (from the commutation relations 2.2.11 we can always put the expression normal ordered up to a constant). The algebra 2.1.33 itself may suffer an anomaly[2] for the case of $m = -n$.

The possible correction on L_0 appears in the form

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n + a^X \tag{2.2.12}$$

We will derive the value for a^X. The anomaly of the algebra will just be stated and the derivation is left for the section 3.5, even though we could already calculate it with some effort

$$[L_m, L_n] = (m-n)\,L_{m+n} + \delta_{m+n}\frac{D}{12}\left(m^3 - m\right) \tag{2.2.13}$$

This expression is used to determine a^X. We have simply $[L_1, L_{-1}] = 2L_0$. A direct computation of the expressions for $L_{\pm 1}$ is possible but it is simpler to use the fact that $2L_0\,|0,0\rangle = 2a^X\,|0,0\rangle$ where $|0,0\rangle$ is the state annihilated by all the oscillator modes α_{-n} with $n \geq 0$[3]. Working with $L_{\pm 1}$we have terms $\alpha_n \cdot \alpha_{1-n}$ and $\alpha_m \cdot \alpha_{-1-m}$. For the result to be non-zero we have to take all of them smaller then zero or, if one is positive, it needs another symmetric term. It is clear that is not possible for them to be all negative but for the cases $m = -n$ and $n = 1 + m$ we find two symmetric pairs. Nonetheless they vanish because of the difference $L_1 L_{-1} - L_{-1} L_1$. With this we conclude that $a^X = 0$.

Even though we determined the quantum expression for L_0, in future expressions involving L_0 will always have a constant a in the form $L_0 \rightarrow L_0 - a$ to be fixed for consistency of the theory. One might think that we have just done that and the theory will be (hopefully) consistent for $a = 0$. This is not true. The Virasoro algebra is consistent for $a^X = 0$ but the theory is not consistent for $a = 0$. This will be explicit already in light cone quantization. In other words we can say that a must be introduced every time we find the sum of oscillator modes in the form $\sum_{n>0} \alpha_{-n} \cdot \alpha_n$. This constant can not be incorporated in L_0 in order to maintain the structure of Virasoro algebra.

[2]By anomaly we mean that for the case $m = -n$ the commutation relation has an extra term that was absent on the classical theory. We will always call it anomaly.

[3]The spectrum is discussed at the end of the section.

2.2.1.2. Lorentz invariance in Light-Cone quantization

As mentioned earlier when we switched for light-cone coordinates the Lorentz invariant look of the theory was lost. Now we have to guarantee Lorentz invariance by hand, having two free parameters to fix: The dimension D and the constant a. In a covariant quantization the angular momentum operators obtained form 2.1.37 obey the Poincaré algebra (note that there is no order ambiguity in 2.1.38)

$$\left[M^{\mu\nu}, M^{\alpha\beta}\right] = i\eta^{\mu\alpha}M^{\nu\beta} + i\eta^{\nu\beta}M^{\mu\alpha} - i\eta^{\nu\alpha}M^{\mu\beta} - i\eta^{\mu\beta}M^{\nu\alpha} \tag{2.2.14}$$

The problem comes from the fact that in the light-cone gauge α_n^- are fixed trough 2.2.9, and, in general, the commutation relation do not respect the Poincaré algebra due to anomalies. This will emerge in the calculation of $\left[M^{i-}, M^{j-}\right]$ that must be zero. The result is obtained by direct computation

$$\left[M^{i-}, M^{j-}\right] = \frac{2}{(p^+)^2}\sum_{m>0}\left[\left(\frac{D-2}{24}-1\right)n + \frac{1}{n}\left(a-\frac{D-2}{24}\right)\right]\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right) + (\alpha \leftrightarrow \tilde{\alpha}) \tag{2.2.15}$$

The constant a and the dimension of the theory D are then fixed guaranteeing the Lorentz invariance

$$\begin{gathered} a = 1 \\ D = 26 \end{gathered} \tag{2.2.16}$$

In light-cone quantization the values of a and D are fixed by requiring Lorentz invariance. We can now proceed to study the spectrum.

2.2.1.3. The spectrum of the string

Having fixed these parameters we can now sketch the spectrum of the string. Going back to equation 2.1.34 we obtain the quantum correction by introduction of a:

$$M^2 = s^2\frac{1}{\alpha'}\left(\sum_{m>0}\alpha_m\cdot\alpha_{-m} - 1\right) = s^2\frac{1}{\alpha'}\left(\sum_{m>0}\tilde{\alpha}_m\cdot\tilde{\alpha}_{-m} - 1\right) \tag{2.2.17}$$

The ground state $|0,p\rangle$ is a **tachyon** (negative mass squared).

$$M^2 = -s^2\frac{1}{\alpha'} \tag{2.2.18}$$

This is not a very serious problem. Consider a quantum field theory with a field ϕ that gives rise to a particle. If $V(\phi)$ is the potential term on the action then the mass squared is simply given by

$$M^2 = \left.\frac{d^2V}{d\phi^2}\right|_{\phi=0} \tag{2.2.19}$$

telling us that we are expanding the potential at the maximum of the tachyon field. The question of existence of a minimum of the tachyon potential still has no answer yet.

Higher excited states are obtained by acting with the creation operators on the fundamental state. For closed string, for example, there are $(D-2)^2 = 24^2$ one excited states:

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}\left|0,p\right\rangle, \ \alpha^i_{-1}\left|0,p\right\rangle \tag{2.2.20}$$

These are respectively for closed and open strings.

Because of the matching condition 2.2.17 we cannot act with just one creation operator in the closed string case, but with two: one for each right and left mode. These are the massless particles. Let us focus on the closed string case where we find 24^2 particle sates. We can decompose this sates transforming in $SO(24)$ in three irreducible representations to which we associate three massless fields:

- A symmetric traceless field $G_{\mu\nu}$
- An anti-symetric field $B_{\mu\nu}$
- A scalar field (trace) Φ

The second and third fields are known by anti-symetric field and dilaton. The most interesting one is the first corresponding to a massless field with spin two (we can verify the spin simply by applying the angular momentum operator for zero momentum). As far as we know this is the graviton which may constitutes a surprise. We took Polyakov action in 26 dimensional flat space with no dynamical background and yet the graviton emerged.

Higher excited states are obtained in the standard way and all of them will be massive.

There is subtlety that is worth to point. When fixing the dimension D and the constant a we guaranteed Lorentz invariance but now all of these states must fit either on $SO(D-2)$ representation if they are massless, or in $SO(D-1)$ if they are massive. It is possible to show that the parameters are just right for this to happen.

2.2.2. Old Covariant Quantization

We sketch now a covariant approach to quantization. Here we pass directly to quantum mechanics using the same recipe $[...]_{P.B.} \to -i[...]$. From our previous work we can suspect the appearance of problems due to the existence of a residual gauge freedom not completely fixed. The commutation relations are

$$\begin{gathered}[x^\mu, p^\nu] = i\eta^{\mu\nu} \\ [\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\end{gathered} \tag{2.2.21}$$

The Fock space constructed by applying the raising operators on the vacuum it is not the physical space since is not positive definite. This happens because of the wrong sign on the commutation relation provided by η^{00} [4]. The physical space must be a subspace of the total Fock space where

[4]Explicitly we can use the commutation relation to see that $\left\langle 0;p\right|\alpha^0_1, \alpha^0_{-1}\left|0;p\right\rangle = \left\langle 0;p'\right|\alpha^0_1, \left(\alpha^0_1\right)^\dagger\left|0;p\right\rangle = -\delta^{(D)}\left(p-p'\right)$ which implies that $\left(\alpha^0_1\right)^\dagger\left|0;p\right\rangle$ has negative norm.

ghosts like these do not appear. Classically there is still the condition 2.1.31 to impose. Quantum mechanically we may think that the solution is to say that for a physical state $L_m \left|\text{Phys}\right\rangle = 0$ for all m but this is too strong and will lead to inconsistencies. A well structured theory arises from physical states conditions:

$$(L_m - a\delta_m)\left|\text{Phys}\right\rangle = 0,\ m \geq 0 \tag{2.2.22}$$

This is not the whole story yet. These conditions imply that any state of the form $\left|\chi\right\rangle = \sum_{m>0} L_{-m}\left|\chi_m\right\rangle$ is orthogonal to all physical states (because $L_{-m} = L_m^\dagger$). Such states are named spurious. A physical spurious state is called null. Every two physical states that differ from each other from a null state will always present the same (relevant) scalar products and will be physically indistinguishable. As a consequence one should define the physical states belonging to classes of equivalence where every two physical states belong to the same class if they differ from each other by a null state. With these observations it is possible to recover the physical space from light-cone quantization.

3. Conformal field theory

Conformal field theory has a wide range of application outside string theory. Particularly it plays a fundamental role in the study of critical phenomena in statistical physics. A conformal transformation is defined is defined as a change of coordinates $\sigma^\alpha \to \sigma'^\alpha\left(\sigma^\beta\right)$ such that the metric changes by

$$g_{\alpha\beta} \longrightarrow \Omega\left(\sigma\right) g_{\alpha\beta}\left(\sigma\right) \tag{3.0.1}$$

A conformal field theory (CFT) is a field theory invariant under conformal transformations. Such theory must look the same at all length scales.

In the present case we are working in a dynamical background where conformal transformations are residual gauge transformations (diffeomorphism+Weyl)[1].

In what follows we change the coordinates of the Polyakov action 2.1.11 to the complex coordinates

$$z = \sigma^1 + i\sigma^2,\ \bar{z} = \sigma^1 - i\sigma^2 \Rightarrow g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2},\ g_{zz} = g_{\bar{z}\bar{z}} = 0 \tag{3.0.2}$$

and use the path integral formalism. We will in general assume closed strings. For open strings much of what is said here is also applicable with minor changes.

It is useful to recall the divergence theorem in complex coordinates

$$\int_R d^2z \left(\partial v^z + \bar{\partial} v^{\bar{z}}\right) = i \oint_{\partial R} \left(v^z d\bar{z} - v^{\bar{z}} dz\right) \tag{3.0.3}$$

v is a general vector field, the derivatives are implicitly taken with respect to z and $\bar{z}$ and the last integral circles the region R counterclockwise.

The Polyakov action in these coordinates is

$$S = \frac{1}{2\pi\alpha'} \int d^2z \partial X^\mu \bar{\partial} X_\mu \tag{3.0.4}$$

and the equations of motion take the form

$$\partial\bar{\partial} X^\mu = 0 \tag{3.0.5}$$

In the quantum theory this is valid as an operator equation. We will now apply the path integral formalism. In section 4 we will discuss in detail the path integral quantization. In the path integral formalism the mean value of an operator $\mathcal{O}$, depending on the fields ϕ, is calculated inserting the

[1] In counterpart, if we were working on a theory with a fixed background, the conformal transformation would be a physical symmetry indeed.

operator in the path integral

$$\langle \mathcal{O} \rangle = \int D\phi \mathcal{O} e^{-S} \tag{3.0.6}$$

Here it will be useful the fact that the path integral of a total derivative is zero. Using this on $\int DXe^{-S}$ we conclude that

$$\left\langle \frac{\delta S}{\delta X_\mu (z, \bar{z})} \right\rangle = 0 \tag{3.0.7}$$

Particularizing this result for the Polyakov action:

$$\left\langle \partial\bar{\partial} X_\mu (z, \bar{z}) \right\rangle = 0 \tag{3.0.8}$$

By the same process we obtained the equation of motion for X^μ, we can consider $\int DX\mathcal{F}e^{-S}$ to obtain:

$$\left\langle \partial\bar{\partial} X_\mu (z, \bar{z}) \mathcal{F} \right\rangle = 0 \tag{3.0.9}$$

Here there was assumed that $\mathcal{F}$ had no dependence on z and $\bar{z}$. Considering: $\mathcal{F} = X^\nu (z', \bar{z}')$ we get:

$$\left\langle \partial\bar{\partial} X^\mu (z, \bar{z}) X^\nu (z', \bar{z}') \right\rangle = -\pi\alpha' \eta^{\mu\nu} \left\langle \delta^{(2)} (z - z', \bar{z} - \bar{z}') \right\rangle \tag{3.0.10}$$

Or even with $\mathcal{F} \to X^\nu (z', \bar{z}') \mathcal{F}$ (again without z and $\bar{z}$ dependence on $\mathcal{F}$):

$$\left\langle \partial\bar{\partial} X^\mu (z, \bar{z}) X^\nu (z', \bar{z}') \mathcal{F} \right\rangle = -\pi\alpha' \eta^{\mu\nu} \left\langle \delta^{(2)} (z - z', \bar{z} - \bar{z}') \mathcal{F} \right\rangle \tag{3.0.11}$$

In what follows we will omit the brackets $\langle\rangle$since these equations hold as operator equations.

3.1. Conformal Group in two dimensions

In this section we make an overview of the conformal group in two dimensions. By definition:

$$g'^{\mu\nu} = \left(\frac{\partial w^\mu}{\partial z^\alpha}\right) \left(\frac{\partial w^\mu}{\partial z^\alpha}\right) g^{\alpha\beta} = \Lambda (z) g^{\mu\nu} \tag{3.1.1}$$

In two dimensions this can be written explicitly:

$$\begin{cases} \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \\ \\ \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1} = 0 \end{cases} \tag{3.1.2}$$

We can use a quick trick to solve this system of equations if we write this system as:

$$\begin{cases} a_0^2 + a_1^2 = b_0^2 + b_1^2 \\ \\ a_1 b_0 + a_0 b_1 = 0 \end{cases}$$

This is equivalent to consider two vectors $(a_0, -a_1)$ and (b_0, b_1) requiring that they have the same length and at the same time the matrix formed by them has a null determinant. This latter condition means they are linear dependent, so: $(a_0, -a_1) = \pm (b_0, b_1)$. Then:

$$\begin{cases} \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \text{ and } \frac{\partial w^0}{\partial z^1} = -\frac{\partial w^1}{\partial z^0} \\ \\ \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \text{ and } \frac{\partial w^0}{\partial z^1} = \frac{\partial w^1}{\partial z^0} \end{cases} \tag{3.1.3}$$

The fact that these are the Cauchy-Riemann equations for (anti)holomorphic functions motivates us to change to complex variables. Then:

$$\begin{array}{ccc} z = z^0 + iz^1 & & z^0 = \frac{z+\bar{z}}{2} \\ & \Leftrightarrow & \\ \bar{z} = z^0 - iz^1 & & z^1 = \frac{z-\bar{z}}{2i} \end{array} \tag{3.1.4}$$

and the derivatives:

$$\begin{array}{ccc} \partial = \frac{1}{2}\partial_0 - \frac{i}{2}\partial_1 & & \partial_0 = \partial + \bar{\partial} \\ & \Leftrightarrow & \\ \bar{\partial} = \frac{1}{2}\partial_0 + \frac{i}{2}\partial_1 & & \partial_1 = i\left(\partial - \bar{\partial}\right) \end{array} \tag{3.1.5}$$

In this coordinate system the metric is:

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{3.1.6}$$

The equations for the holomorphic and antiholomorphic case are respectively:

$$\bar{\partial} w\left(z, \bar{z}\right) = 0 \qquad \partial \bar{w}\left(z, \bar{z}\right) = 0 \tag{3.1.7}$$

We conclude that the conformal group in two dimensions is the set of all analytical maps along with the composition of functions. Since there is a bijective relation at some neighborhood between the set of analytical functions and a Laurent series, the group is infinite dimensional (regarding the coefficients of the series).

It is important to note that $\bar{z}$ is not to be regarded as a complex conjugate of z but an independent complex coordinate. However we did not extend the physical space to a four dimensional space, and we should consider the physical space as the submanifold defined by $z^* = \bar{z}$.

For now all the statements are local, so the transformations do not need to be invertible and even defined everywhere. In order to have a group this must not be the case. This states the difference between local and global transformations.

The form of the global conformal transformations are rather simple. To see their form we note that given such a transformation $f(z)$, the only acceptable singularities are poles, so f must be a ratio of polynomials without common zeros:

$$f(z) = \frac{P(z)}{Q(z)} \tag{3.1.8}$$

If P as several zeros at different points then f is not invertible. If P as a multiple zero at z_0, a small neighborhood of this point is sent n times around z_0. Explicitly if we have to solve $z^n = w$ in polar coordinates, we would have $|z|^n e^{in\theta} = |w| e^{i\alpha}$:

$$\begin{cases} |z|^n = |w| \\ \\ n\theta = 2\pi m + \alpha \end{cases}$$

The last condition allows n possible solutions where θ is in the interval $[0, 2\pi[$, so only $n = 0, 1$ is suitable. The same kind of argument can be applied to Q if we regard the behavior at infinity. Then:

$$\begin{aligned} f(z) &= \tfrac{az+b}{cz+d} \\ \\ ad - bc &= 1 \end{aligned} \tag{3.1.9}$$

To have different zeros $ad - bc \neq 0$. Since an overall scale does not change the map, conventionally the normalization is set to be 1.

In this framework we will be interested in local transformations.

3.2. The operator product expansion - OPE

The operator product expansion (OPE) is an important tool of conformal field theory that describes what happens when two local operators approach each other. Before we describe what is exactly the OPE and how it can be obtained, it is useful to define “normal ordering” in the following way:

$$: \mathcal{F} :\equiv \exp\left(\frac{\alpha'}{4}\int d^2z_1 d^2z_2 \log|z_1 - z_2|^2 \frac{\delta}{\delta X^\mu(z_1, \bar{z}_1)} \frac{\delta}{\delta X_\mu(z_2, \bar{z}_2)}\right)\mathcal{F} \tag{3.2.1}$$

This consists of an expansion in which for the nth order term we will have $2n$ functional derivatives acting on $\mathcal{F}$ with $2n$ integrals. In particular:

$$: X^\mu(z, \bar{z}) := X^\mu(z, \bar{z}) \tag{3.2.2}$$

$$: X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') := X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') + \frac{\alpha'}{2}\eta^{\mu\nu}\log|z - z'|^2 \tag{3.2.3}$$

The main point of this definitions is the fact that:

$$\partial\bar{\partial} : X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') := 0 \tag{3.2.4}$$

which can be easily proved using equation 3.0.10 and $\partial\bar{\partial}\log|z|^2 = 2\pi\delta^{(2)}(z, \bar{z})$. This equation clearly holds for $z \neq 0$ and we can find the 2π factor by integration. We put $\partial\bar{\partial}\log|z|^2 = \bar{\partial}\frac{1}{z} + \partial\frac{1}{\bar{z}}$ on the divergence theorem 3.0.3 and use the residues theorem.

From 3.2.3 we can also get the normal product of the fields with derivatives:

$$
\begin{aligned}
&:\partial X^{\mu}(z,\bar{z})\, X^{\nu}(z',\bar{z}') := \partial X^{\mu}(z,\bar{z})\, X^{\nu}(z',\bar{z}') + \tfrac{\alpha'}{2}\eta^{\mu\nu}\tfrac{1}{z-z'} \\
&:\bar{\partial} X^{\mu}(z,\bar{z})\, X^{\nu}(z',\bar{z}') := \bar{\partial} X^{\mu}(z,\bar{z})\, X^{\nu}(z',\bar{z}') + \tfrac{\alpha'}{2}\eta^{\mu\nu}\tfrac{1}{\bar{z}-\bar{z}'}
\end{aligned}
\tag{3.2.5}
$$

This result will follow in the same way for an arbitrary number of fields.

Applying the definition of normal product to the product of two general operators $:\mathcal{FG}:$ gives the result:

$$
:\mathcal{F}::\mathcal{G}:=:\mathcal{FG}:+:\sum \text{cross}-\text{contractions}: \tag{3.2.6}
$$

By contraction we mean replacing each pair of fields $X^{\mu}(z,\bar{z})$ and $X^{\nu}(z',\bar{z}')$ by $-\frac{1}{2}\alpha'\eta^{\mu\nu}\log|z-z'|^2$. By cross-contractions we mean that one field must come from the operator $\mathcal{F}$ and the other from $\mathcal{G}$.

We move now to the OPE. The main idea is that two operators at two nearby points can be written as a string of local operators at one of the points. An OPE takes the general form

$$
\mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\bar{w}) = \sum_k C_{ij}^{k}(z-w,\bar{z}-\bar{w})\,\mathcal{O}_k(w,\bar{w}) \tag{3.2.7}
$$

where k runs over the set of all local operators and $C_{ij}^{k}(z-w,\bar{z}+\bar{w})$ are a set of functions depending only on the separation of the two points. They are decreasingly smaller but they may be singular as $z\to w$. Again, this means:

$$
\langle \mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\bar{w})\,\mathcal{F}\rangle = \sum_k C_{ij}^{k}(z-w,\bar{z}+\bar{w})\,\langle \mathcal{O}_k(w,\bar{w})\,\mathcal{F}\rangle
$$

The separation between z and w must be small when compared to any point of $\mathcal{F}$.

In what follows we will derive the OPE for the X^{μ} theory. From 3.2.4 we see that the normal product is a harmonic function. From the theory of complex analysis we know that such a function is a sum of an holomorphic and an antiholomorphic function, by which they can be Taylor expanded. Using these on equation 3.2.3:

$$
\begin{aligned}
X^{\mu}(z,\bar{z})\,X^{\nu}(z',\bar{z}') = -\tfrac{\alpha'}{2}\eta^{\mu\nu}\log|z-z'|^2 \\
+\sum_{k=1}^{+\infty}\tfrac{1}{k!}\left[(z-z')^k :\partial^k X^{\mu}(z,\bar{z})\,X^{\nu}(z',\bar{z}')\,|_{z=z'}: + (\bar{z}-\bar{z}')^k :\bar{\partial}^k X^{\mu}(z,\bar{z})\,X^{\nu}(z',\bar{z}')\,|_{\bar{z}=\bar{z}'}:\right]
\end{aligned}
\tag{3.2.8}
$$

This has the form of the OPE 3.2.7 and it illustrates how to obtain the OPE for general operators: for any product of two operators (say $\mathcal{F}$ and $\mathcal{G}$) we can apply normal ordering from equation 3.2.1. The first term of the series is the product itself, $\mathcal{FG}$, that can be isolated. The term $:\mathcal{FG}:$ constitutes only regular termswith contractions that will give the divergent terms.

3.3. Noether theorem and Ward Identities

Here we discuss the consequences of symmetries of the action. Let ϕ_α be general fields, the transformation $\phi_\alpha \to \phi_\alpha^{'}$ is a symmetry if the product of the path integral measure and the weight e^{-S} is invariant:

$$D\phi^{'} e^{-S^{'}} = D\phi e^{-S} \tag{3.3.1}$$

We abbreviated $S[\phi^{'}] \equiv S^{'}$. Consider an infinitesimal change of variables in the light of this symmetry

$$\phi_\alpha^{'} (\sigma) = \phi_\alpha (\sigma) + \delta\phi_\alpha (\sigma) \tag{3.3.2}$$

and also the transformation

$$\phi_\alpha^{'} (\sigma) = \phi_\alpha (\sigma) + \rho (\sigma) \delta\phi_\alpha (\sigma) \tag{3.3.3}$$

which in general is not a symmetry, but it would be if $\rho(\sigma)$ was constant. Then equation 3.3.1 must change by something proportional to $\partial_\alpha \rho$, that, in the most general way is:

$$D\phi^{'} e^{-S^{'}} = D\phi e^{-S} \left(1 + \frac{i\epsilon}{2\pi} \int d^d\sigma \sqrt{g} j^a (\sigma) \partial_a \rho (\sigma) + O\left(\epsilon^2\right)\right) \tag{3.3.4}$$

This factor, j^a, is a local operator coming from variations of the measure and of the weight. Despite of this variation, a general transformation just makes a redefinition of a dummy variable, so:

$$\begin{gathered} \int D\phi^{'} e^{-S^{'}} = \int D\phi e^{-S} \\ \Rightarrow \int D\phi e^{-S} \int d^d\sigma \sqrt{g} \partial_a j^a (\sigma) \rho (\sigma) = 0 \end{gathered} \tag{3.3.5}$$

In this step, we also integrated by parts to obtain the divergence of j^a. Having to hold for all ρ, we obtain the Noether theorem:

$$\langle \partial_a j^a \rangle = 0 \tag{3.3.6}$$

Now let R be some region. We particularize:

$$\rho (\sigma) = \begin{cases} 1, & \text{if } \sigma \in R \\ 0, & \text{if } \sigma \notin R \end{cases} \tag{3.3.7}$$

Consider $\sigma_0 \in R$ and $\int D\phi e^{-S} \mathcal{O} (\sigma_0)$ where $\mathcal{O}(\sigma_0)$ is a local operator. Proceeding in the same way as before:

$$\delta \mathcal{O} (\sigma_0) - \frac{i\epsilon}{2\pi} \int_R d^d\sigma \sqrt{g} \partial_a j^a (\sigma) \mathcal{O} (\sigma_0) = 0 \tag{3.3.8}$$

These are the Ward identities and they tell us how the operators transform due to a symmetry of the action.

The Ward identities can be presented in another way. If we consider a vector n_a normal to the surface R, then, by the divergence theorem:

$$\int_{\partial R} dA n_a j^a \mathcal{O}(\sigma_0) = \int_R d^d\sigma \sqrt{g} \partial_a j^a \mathcal{O}(\sigma_0) = \frac{2\pi}{i\epsilon} \delta\mathcal{O}(\sigma_0) \tag{3.3.9}$$

Particularizing to the free theory in two dimensions, making the change of variables to the complex coordinates:

$$\oint_{\partial R} \left(j dz - \bar{j} d\bar{z} \right) \mathcal{O}(\sigma_0) = \frac{2\pi}{i\epsilon} \delta\mathcal{O}(\sigma_0) \tag{3.3.10}$$

While Noether's theorem guarantees the existence of a conserved current for each symmetry of the system, the Ward identities indicates how the operators changes and will put constraints on the OPE.

We move on to the application of these results to particular cases of the theory. Consider the world-sheet translation symmetry $\delta\sigma^a = \epsilon v^a$. Following the Noether procedure we arrive at the conserved current

$$j_a = i v^b T_{ab} \tag{3.3.11}$$

being T_{ab} the energy-momentum tensor that was calculated earlier in 2.1.15. In this formalism the products within operators are defined with normal ordering, so what we really have is:

$$T_{ab} = -\frac{1}{\alpha'} : \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} g_{ab} \partial_c X^\mu \partial^c X_\mu \right) : \tag{3.3.12}$$

There is an ambiguity in this renormalization: we may take an additive constant times g_{ab}. Here we take this constant to zero.

Earlier we saw that the trace of the energy momentum tensor was zero. In fact, at a classical level this is true for all conformal theories. If we consider a scale transformation $\delta g_{ab} = \epsilon g_{ab}$, then:

$$0 = \delta S = \int d^2\sigma \frac{\delta\mathcal{L}}{\delta g_{ab}} \delta g_{ab} \propto \int d^2\sigma T^{ab} g_{ab} \propto \int d^2\sigma T^a_{\ a} \tag{3.3.13}$$

$$T^a_{\ a} = 0 \tag{3.3.14}$$

In complex coordinates we will write

$$\begin{cases} T_{z\bar{z}} = 0 \\ \bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0 \end{cases} \Rightarrow \begin{cases} T_{zz} \equiv T(z) \\ T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}) \end{cases} \tag{3.3.15}$$

For the free theory

$$\begin{cases} T(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \\ \bar{T}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : \end{cases} \tag{3.3.16}$$

The traceless property of this tensor implies a larger symmetry, in fact, an "infinite source" of symmetries. To see this, consider an infinitesimal general transformation:

$$z' = z + \epsilon\left(z, \bar{z}\right)$$
$$\bar{z}' = \bar{z} + \bar{\epsilon}\left(z, \bar{z}\right) \tag{3.3.17}$$

If the theory was to be coupled with gravity it would be diffeomorphism invariant and the variation of the action would be null. Being in the free theory we have just a change on the action due to the change on the derivatives and on the fields. The resultant variation is then symmetric to the pure change of the metric:

$$\delta S = -\int d^2\sigma \tfrac{\delta \mathcal{L}}{\delta g^{ab}} \delta g^{ab} \propto \int d^2 z T^{ab} \partial_a \delta z_b = \int d^2 z T^{ab} \partial_a \delta z_b = \int d^2 z \left(T(z) \partial^z \epsilon\left(z, \bar{z}\right) + \bar{T}(\bar{z}) \partial^{\bar{z}} \bar{\epsilon}\left(z, \bar{z}\right)\right)$$
$$= 2\int d^2 z \left(T(z) \bar{\partial} \epsilon\left(z, \bar{z}\right) + \bar{T}(\bar{z}) \partial \bar{\epsilon}\left(z, \bar{z}\right)\right) \tag{3.3.18}$$

We immediately see that if ϵ is holomorphic and $\bar{\epsilon}$ anti-holomorphic then the variation on the action is zero. This will guarantee the conservation of the currents for any holomorphic $v\left(z\right)$ for which $z' = z + \epsilon v\left(z\right)$:

$$j\left(z\right) = iT\left(z\right) v\left(z\right)$$
$$\bar{j}\left(\bar{z}\right) = i\bar{T}\left(\bar{z}\right) v^*\left(\bar{z}\right) \tag{3.3.19}$$

These currents are conserved individually.

Using the property of equation 3.2.6 we can write the OPE for $T(z)X^\mu\left(0,0\right)$ and $\bar{T}(\bar{z})X^\mu\left(0,0\right)$:

$$T(z)X^\mu\left(0,0\right) = -\tfrac{1}{\alpha'} : \partial X^\alpha\left(z, \bar{z}\right) \partial X_\alpha\left(z, \bar{z}\right) X^\mu\left(0,0\right) : + \eta^{\mu\alpha} \tfrac{1}{z} \partial X_\alpha\left(z, \bar{z}\right) \sim \tfrac{1}{z} \partial X^\mu\left(0,0\right)$$
$$\bar{T}\left(\bar{z}\right) X^\mu\left(0,0\right) = -\tfrac{1}{\alpha'} : \bar{\partial} X^\alpha\left(z, \bar{z}\right) \bar{\partial} X_\alpha\left(z, \bar{z}\right) X^\mu\left(0,0\right) : + \eta^{\mu\alpha} \tfrac{1}{\bar{z}} \bar{\partial} X_\alpha\left(z, \bar{z}\right) \sim \tfrac{1}{\bar{z}} \bar{\partial} X^\mu\left(0,0\right) \tag{3.3.20}$$

In the last part we left just the divergent term.

Because $j\left(z\right)$ $\left(\left(\bar{j}\left(\bar{z}\right)\right)\right)$ are (anti)holomorphic the Ward identities 3.3.10 read

$$\operatorname*{Res}_{z \to z_0} j\left(z\right) \mathcal{O}\left(z_0, \bar{z}_0\right) + \operatorname*{\bar{Res}}_{z \to z_0} \bar{j}\left(\bar{z}\right) \mathcal{O}\left(z_0, \bar{z}_0\right) = \frac{1}{i\epsilon} \delta \mathcal{O}\left(z_0, \bar{z}_0\right) \tag{3.3.21}$$

The Ward identities and the OPE allow us to write the variation of the operator X^μ with respect to the infinitesimal transformation $z' = z + \epsilon v\left(z\right)$:

$$\delta X^\mu = -\epsilon v\left(z\right) \partial X^\mu - \epsilon v^*\left(\bar{z}\right) \bar{\partial} X^\mu \tag{3.3.22}$$

The finite transformation is

$$X'^\mu\left(z', \bar{z}'\right) = X^\mu\left(z, \bar{z}\right), \;\; z = f\left(z\right) \tag{3.3.23}$$

We see a wide range of symmetry, a non-trivial statement of the theory.

3.4. Quasi-Primary and Primary Operators

In this section we shall discuss some particular operators called quasi-primary and primary operators.

Because T is holomorphic its OPE with other operator $\mathcal{O}$ must be a Laurent expansion:

$$T(z)\,\mathcal{O}(0,0) = \sum_n \frac{1}{z^{n+1}}\mathcal{O}^{(n)}(0,0) \tag{3.4.1}$$

where $\mathcal{O}^{(n)}(0,0)$ are to be determined by the conformal transformations. In fact, from 3.3.21 we get:

$$\delta\mathcal{O}(z_0,\bar{z}_0) = -\epsilon\sum_n \frac{1}{n!}\left(\partial^n v(z)\,\mathcal{O}^{(n)}(z,\bar{z}) + \bar{\partial}^n v(z)^*\,\bar{\mathcal{O}}^{(n)}(z,\bar{z})\right) \tag{3.4.2}$$

Now let us define quasi-primary and a primary fields. A field ϕ is said to be quasi-primary if under a scaling $w = \lambda z$ it transforms like:

$$\phi'(w,\bar{w}) = \lambda^{-h}\lambda^{-\tilde{h}}\phi(z,\bar{z}) \tag{3.4.3}$$

The pair $\left(h,\tilde{h}\right)$ are known as the weights or conformal dimension of ϕ. A derivative ∂ acting on ϕ increases h by one, while the derivative $\bar{\partial}$ increases $\tilde{h}$ by one. A generalization of this property is obeyed by primary field that under a conformal transformation $w = f(z)$ transforms like:

$$\phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\tilde{h}}\phi(z,\bar{z}) \tag{3.4.4}$$

meaning that in an infinitesimal transformation:

$$\begin{aligned} w &= z + \epsilon v(z) \\ \bar{w} &= \bar{z} + \bar{\epsilon}\bar{v}(\bar{z}) \end{aligned} \tag{3.4.5}$$

we get

$$\begin{aligned} \phi'(w,\bar{w}) &= \left(1 - h\partial\epsilon(z) - \tilde{h}\bar{\partial}\bar{\epsilon}(\bar{z})\right)\phi(z,\bar{z}) \\ \Rightarrow \delta\phi &= -\epsilon\left(v\partial + h\partial v + \bar{v}\bar{\partial} + \tilde{h}\bar{\partial}\bar{v}\right)\phi(z,\bar{z}) \end{aligned} \tag{3.4.6}$$

For linear $v(z) = z$, the transformation is an eigenstate. As a consequence, the quasi-primary operators, are eigenstates of dilatations and rotations. Explicitly $\epsilon = \bar{\epsilon}$ is an infinitesimal dilatation and $\epsilon = -\bar{\epsilon}$ is a rotation, motivating the definitions:

- Dimension: $\Delta = h + \tilde{h}$.
- Spin: $s = h - \tilde{h}$.

3.4.1. Two and three point correlation functions for quasi-primary operators

Given two operators, they have the general OPE 3.2.7, but now, given a conformal transformation $z \to \lambda z$ and $w \to \lambda w$ the functions $C_{ij}^{k}\left(z-w, \bar{z}-\bar{w}\right)$ are forced to obey:

$$\lambda^{-h_i-h_j}\bar{\lambda}^{-\tilde{h}_i-\tilde{h}_j}C_{ij}^{k}\left(z-w,\bar{z}-\bar{w}\right)=\lambda^{-h_k}\bar{\lambda}^{-\tilde{h}_k}C_{ij}^{k}\left(\lambda z-\lambda w,\bar{\lambda}\bar{z}-\bar{\lambda}\bar{w}\right) \tag{3.4.7}$$

Reorganizing we get:

$$C_{ij}^{k}\left(\lambda z-\lambda w,\bar{\lambda}\bar{z}-\bar{\lambda}\bar{w}\right)=\lambda^{-h_i-h_j+h_k}\bar{\lambda}^{-\tilde{h}_i-\tilde{h}_j+\tilde{h}_k}C_{ij}^{k}\left(z-w,\bar{z}-\bar{w}\right) \tag{3.4.8}$$

which fixes the function up to multiplicative constant:

$$C_{ij}^{k}\left(z-w,\bar{z}-\bar{w}\right)=c_{ij}^{k}\left(z-w\right)^{h_k-h_i-h_j}\left(\bar{z}-\bar{w}\right)^{\tilde{h}_k-\tilde{h}_i-\tilde{h}_j} \tag{3.4.9}$$

The form of the two and three point correlation functions is fixed for quasi-primary fields. First, by translation invariance we must have:

$$\left\langle \mathcal{O}_i\left(z,\bar{z}\right)\mathcal{O}_j\left(w,\bar{w}\right)\right\rangle = f_{ij}\left(z-w,\bar{z}-\bar{w}\right) \tag{3.4.10}$$

for some function f. Now, because they are quasi-primary fields:

$$f_{ij}\left(\lambda\left(z-w\right),\bar{\lambda}\left(\bar{z}-\bar{w}\right)\right)=\lambda^{-h_i-h_j}\bar{\lambda}^{-\tilde{h}_i-\tilde{h}_j}f_{ij}\left(z-w,\bar{z}-\bar{w}\right) \tag{3.4.11}$$

We can solve this deriving in order to the parameter and making $\lambda = 1$, obtaining the differential equation:

$$\begin{cases} x\frac{\partial f_{ij}(x,y)}{\partial x} = -\left(h_i+h_j\right)f_{ij}\left(x,y\right) \\ \\ y\frac{\partial f_{ij}(x,y)}{\partial y} = -\left(\tilde{h}_i+\tilde{h}_j\right)f_{ij}\left(x,y\right) \end{cases} \tag{3.4.12}$$

The solution is obtained up to a multiplicative constant

$$\left\langle \mathcal{O}_i\left(z,\bar{z}\right)\mathcal{O}_j\left(w,\bar{w}\right)\right\rangle = \frac{c_{ij}}{\left(z-w\right)^{h_i+h_j}\left(\bar{z}-\bar{w}\right)^{\tilde{h}_i+\tilde{h}_j}} \tag{3.4.13}$$

For primary fields, we have to check consistency not only for scaling but for a general conformal transformation . Applying the transformation and using 3.4.4 will impose conditions on the factors c_{ij} namely:

$$\left\langle \mathcal{O}_i\left(z,\bar{z}\right)\mathcal{O}_j\left(w,\bar{w}\right)\right\rangle = \begin{cases} \frac{c_{ij}}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\tilde{h}}} & ,\ \text{if } h_i=h_j=h,\ \tilde{h}_i=\tilde{h}_j=\tilde{h} \\ \\ 0 & ,\ \text{otherwise} \end{cases} \tag{3.4.14}$$

For the correlation function of two operators not to vanish, they must have the same conformal

Operator	Weights $\left(h,\tilde{h}\right)$
X^μ	$(0,0)$
∂X^μ	$(1,0)$
$\bar{\partial} X^\mu$	$(0,1)$
$:\partial X^\mu \bar{\partial} X^\mu:$	$(1,1)$
$:e^{ikX}:$	$\left(\frac{\alpha' k^2}{4},\frac{\alpha' k^2}{4}\right)$

Table 3.1.: Weights of some of the elementary fields.

dimensions.

The same kind of argument fixes three point functions. With some extra effort one can obtain:

$$\begin{aligned}\left\langle \mathcal{O}_i\left(z_i,\bar{z}_i\right)\mathcal{O}_j\left(z_j,\bar{z}_j\right)\mathcal{O}_j\left(z_k,\bar{z}_k\right)\right\rangle &= c_{ijk}\frac{1}{\left(z_i-z_j\right)^{h_i+h_j-h_k}\left(z_i-z_k\right)^{h_i+h_k-h_j}\left(z_j-z_k\right)^{h_j+h_k-h_i}}\\ &\times\frac{1}{\left(\bar{z}_i-\bar{z}_j\right)^{\tilde{h}_i+\tilde{h}_j-\tilde{h}_k}\left(\bar{z}_i-\bar{z}_k\right)^{\tilde{h}_i+\tilde{h}_k-\tilde{h}_j}\left(\bar{z}_j-\bar{z}_k\right)^{\tilde{h}_j+\tilde{h}_k-\tilde{h}_i}}\end{aligned} \tag{3.4.15}$$

3.4.2. The Stress-Energy tensor and Primary Operators

From equations 3.4.2 and 3.4.6 we can determine the coefficients $\mathcal{O}^{(0)}$ $\mathcal{O}^{(1)}$ in the case of a quasi-primary operators, namely:

$$\begin{aligned}\mathcal{O}^{(0)} = \partial\mathcal{O},\ \bar{\mathcal{O}}^{(0)} = \bar{\partial}\mathcal{O}\\ \mathcal{O}^{(1)} = h\mathcal{O},\ \bar{\mathcal{O}}^{(1)} = \tilde{h}\mathcal{O}\end{aligned} \tag{3.4.16}$$

which determines part of the OPE:

$$T\left(z\right)\mathcal{O}\left(0,0\right) = ... + \frac{h\mathcal{O}}{z^2} + \frac{\partial\mathcal{O}}{z} + ... \tag{3.4.17}$$

For primary operators we can determine all the $\mathcal{O}^{(n)}$ for positive n, and conclude that the most singular term is the $\frac{1}{z^2}$:

$$T\left(z\right)\mathcal{O}\left(0,0\right) = \frac{h\mathcal{O}}{z^2} + \frac{\partial\mathcal{O}}{z} + ... \tag{3.4.18}$$

Comparing this with the OPE 3.3.20 we find that X^μ is a primary operator of weights $(0,0)$. We can write then the Table 3.1 with the elementary weights.

We will now prove the last line of the table, caughting the $\frac{1}{z^2}$ term of the OPE $T\left(z\right):e^{ikX(w)}:$. We have

$$T\left(z\right):e^{ikX(w)}: = -\sum_n \frac{1}{\alpha'}\frac{(i)^n}{n!}:\partial X^\mu\left(z\right)\partial X_\mu\left(z\right):\left(k\cdot X\left(w,\bar{w}\right)\right)^n: \tag{3.4.19}$$

From 3.2.5 we see that the $\frac{1}{z^2}$ terms are obtained by the two cross contractions, these are the terms

$$-\sum_n \frac{1}{\alpha'}\frac{(i)^n}{n!}\left(\frac{\alpha'}{2}\right)^2\frac{n\left(n-1\right)}{\left(z-w\right)^2}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\lambda}k^\alpha k^\sigma\eta^{\mu\beta}\eta^{\nu\lambda}:\left(k\cdot X\left(w,\bar{w}\right)\right)^{n-2}: \tag{3.4.20}$$

Rearranging

$$\frac{1}{(z-w)^2}\frac{k^2\alpha'}{4}\sum_n\frac{(i)^{n-2}}{(n-2)!}:(k\cdot X(w,\bar{w}))^{n-2}:=\frac{1}{(z-w)^2}\frac{k^2\alpha'}{4}:e^{ikX(w)}: \tag{3.4.21}$$

From expression 3.4.17 we conclude

$$h=\frac{\alpha' k^2}{4} \tag{3.4.22}$$

The case of $\tilde{h}$ is, in all aspects, the same.

Another quantity that we may calculate, is the OPE of the T with itself. It is easy to list all the possible cross contractions

$$\begin{gathered}T(z)\,T(w)=\tfrac{1}{\alpha'^2}:\partial X^\mu(z)\,\partial X_\mu(z):\partial X^\alpha(w)\,\partial X_\alpha(w):\\ =\tfrac{\eta_{\mu\nu}\eta_{\alpha\beta}}{\alpha'^2}\left(\tfrac{\alpha'}{2}\right)^2\tfrac{2\eta^{\mu\alpha}\eta^{\nu\beta}}{(z-w)^4}+\tfrac{\eta_{\mu\nu}\eta_{\alpha\beta}}{\alpha'^2}\tfrac{\alpha'}{2}\tfrac{4\eta^{\nu\beta}}{(z-w)^2}:\partial X^\mu(z)\,\partial X^\alpha(w):+\tfrac{1}{\alpha'^2}:\partial X^\mu(z)\,\partial X_\mu(z)\,\partial X^\alpha(w)\,\partial X_\alpha(w):\\ =\tfrac{D/2}{(z-w)^4}+\tfrac{2}{(z-w)^2}T(w)+\tfrac{1}{z-w}\partial T(w)+\text{Non singular terms}\end{gathered} \tag{3.4.23}$$

The presence of the $\frac{1}{z^4}$ term, according with what we have seen before, tells us that $T(z)$ is not a primary operator but rather a quasi-primary operator with weights $(2,0)$. This expression also puts on evidence the non-tensor character of T. By equation 3.4.2 one concludes that T transforms like

$$\delta T=-\epsilon\left(-\frac{D}{12}\partial^3 v-2\partial vT-v\partial T\right) \tag{3.4.24}$$

The finite transformation is

$$(\partial z')^2\,T'(z')=T(z)-\frac{D}{12}\left(\frac{2\partial^3 z'\partial z'-3\left(\partial^2 z'\right)^2}{2\left(\partial z'\right)^2}\right) \tag{3.4.25}$$

This can be checked by proving that it reduces to 3.4.24 for infinitesimal transformations and that the composition of transformation is correct.

3.5. The Virasoro Algebra

We now return to the Virasoro Algebra. We recall that the two dimensions correspond to a time coordinate $-\infty<\tau<+\infty$ and a spacial coordinate with periodicity $\sigma\sim\sigma+2\pi$. This forms an infinite cylinder. It is useful to work on complex coordinates. The first natural choice is the one we have bee using

$$w=\sigma+i\tau \tag{3.5.1}$$

It is possible to map this cylinder to all the complex plane through another possible choice of coordinates

$$z=e^{-iw} \tag{3.5.2}$$

When one works on the cylinder a given state its defined at definite time representing then a slice of the cylinder. Their evolution is governed by the Hamiltonian

$$H = \partial_t \tag{3.5.3}$$

This equal time slices, when mapped to the plane, will be circles around the origin. The radius of these circles increase with the time coordinate. The Hamiltonian will transform to be now the dilaton operator

$$D = z\partial + \bar{z}\bar{\partial} \tag{3.5.4}$$

This is known as radial quantization.

Because of (anti)holomorphism we can make the Laurent expansions on the plane

$$T_z\left(z\right) = \sum_m \frac{L_m}{z^{m+2}},\ \bar{T}_z\left(\bar{z}\right) = \sum_m \frac{\bar{L}_m}{\bar{z}^{m+2}} \tag{3.5.5}$$

which can be inverted using the residues theorem for a contour C circling the origin counterclockwise. This coefficients are known as the Virasoro generators.

$$L_n = \frac{1}{2\pi i}\oint_C dz T_z\left(z\right) z^{n+1},\ \bar{L}_n = \frac{1}{2\pi i}\oint_C d\bar{z}\bar{T}_z\left(\bar{z}\right)\bar{z}^{n+1} \tag{3.5.6}$$

On the cylinder the expansions will be obtained by a transformation of coordinates:

$$T_w\left(w\right) = \sum_m T_m e^{imw},\ \bar{T}_w\left(\bar{w}\right) = \sum_m \bar{T}_m e^{-imw} \tag{3.5.7}$$

The relation between the L_m and the T_m are obtained by using the expression for the transformation 3.4.25

$$T_m = L_m - \delta_m \frac{D}{24},\ \bar{T}_m = \bar{L}_m - \delta_m \frac{D}{24} \tag{3.5.8}$$

The L_n are the conserved charges for the transformation $\delta z = z^{n+1}$. Indeed, as we saw previously the currents 3.3.19 are conserved and taking C to be a circle on the plane the integrations on 3.5.6 are made at fixed time running trough all the spacial coordinate on the cylinder. This result was obtained taking into account one particular CFT since we used the Polyakov action to obtain the stress-energy tensor in 3.3.12. One can show that for a general CFT the OPE 3.4.23 is still valid if we replace the dimension by two constants $D \to c,\ \tilde{c}$ depending if we are working on the OPE of TT or on the $\bar{T}\bar{T}$. These constants are known as the central charges, and this passage suggests that they are measuring something like the degrees of freedom of the CFT. Still, $c,\ \tilde{c}$ are not necessarily integers. In this last calculation, this change will ultimately result in

$$T_m = L_m - \delta_m \frac{c}{24},\ \bar{T}_m = \bar{L}_m - \delta_m \frac{c}{24} \tag{3.5.9}$$

Now we are going to compute the algebra of the $L_m$operators. For this we put 3.5.6 into $[L_m, L_n]$ noting that operator equation are to be viewed under the path integral which automatically puts the

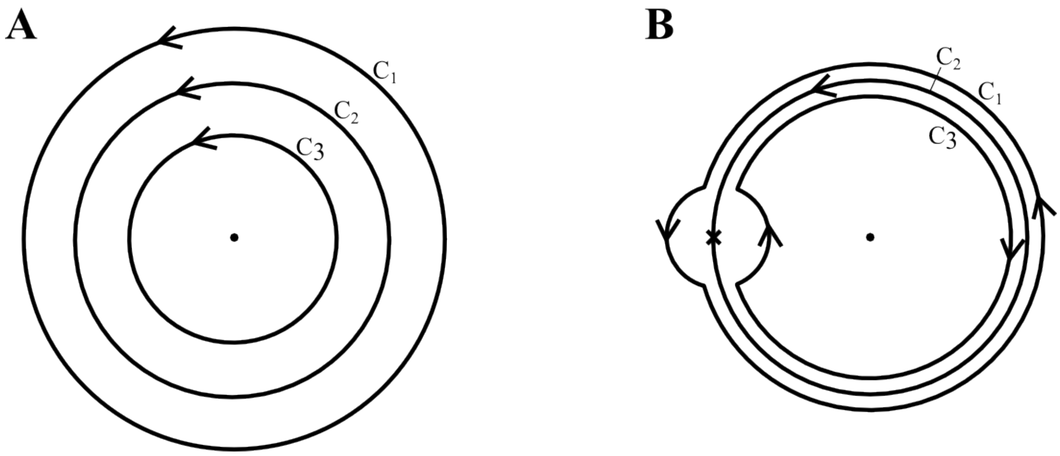


Figure 3.5.1.: The contours

fields into time ordering. Because we are working on the plane, time ordering is now radial ordering and in this sense, the contours should be taken as following and according to the figure 3.5.1 (A).

$$L_m \{C_1\} L_n \{C_2\} - L_n \{C_2\} L_m \{C_3\} \tag{3.5.10}$$

The integrals that we need to compute are

$$[L_m, L_n] = \left(\oint_{C_1} \frac{dx}{2\pi i} \oint_{C_2} \frac{dy}{2\pi i} - \oint_{C_2} \frac{dy}{2\pi i} \oint_{C_3} \frac{dx}{2\pi i} \right) x^{m+1} y^{n+1} T(x) T(y) \tag{3.5.11}$$

Start by the integral on x, deforming homotopically the curves $C_1$and C_3 in a way that they overlap at any point except on the neighborhood of the point y as illustrated in the figure 3.5.1(B), resulting in

$$[L_m, L_n] = \oint_{C_2} \frac{dy}{2\pi i} \operatorname*{Res}_{x \to y} \left(x^{m+1} y^{n+1} T(x) T(y) \right) \tag{3.5.12}$$

This residue is easy to compute using 3.4.23 and expanding x^{m+1} in powers of $(x-y)$.

$$\operatorname*{Res}_{x \to y} \left(x^{m+1} y^{n+1} T(x) T(y) \right) = \frac{c}{12} m \left(m^2 - 1 \right) y^{m+n-1} + 2T(y)(m+1) y^{m+n+1} + \partial T(y) y^{m+n+2} \tag{3.5.13}$$

Putting this into the integral, and doing an integration by parts

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m \left(m^2 - 1 \right) \delta_{m-n} \tag{3.5.14}$$

This is the so called Virasoro algebra with an analogous result for the case of $\tilde{L}_m$. We just proved the relation anticipated in section 2.2.1.1. The anomaly constitutes a problem to be solved.

The contour argument on general charges

We have just argued about contours to obtain the algebra of Virasoro generators using the fact that they are conserved charges and knowing the OPEs of their respective currents. This is always true, and, at this point, it is not hard to see that the OPE of the currents determine the algebra of their charges. Following the same procedure we get

$$[Q_1, Q_2] = \oint_{C_2} \frac{dy}{2\pi i} \operatorname*{Res}_{x\to y} j_1(x)\, j_2(y) \tag{3.5.15}$$

We can even replace one of the charges by another operator. We loose one contour integral, and the one above drops out

$$[Q, \mathcal{O}(y,\bar{y})] = \operatorname*{Res}_{x\to y} j(x)\, \mathcal{O}(y,\bar{y}) = \frac{1}{i\varepsilon}\delta\mathcal{O}(y,\bar{y}) \tag{3.5.16}$$

Where in the last step we went back to equation 3.3.21, considering just an holomorphic conserved current and exhibiting the fact that the charge of the current generates the transformations. The results are obviously analogue for antiholomorphic currents.

3.6. The Weyl anomaly

In a general quantum field theory an anomaly is the breaking of a symmetry that existed classically but does not survive quantization. In this section we explore the breakdown of Weyl invariance.

The Weyl anomaly, also known as a trace anomaly, consists in the the fact that the trace of the stress-energy tensor gets non null in the quantum theory of a curved background. This is a problem since metrics that are connected by a Weyl transformation will have different expectation values for the trace of the stress energy tensor yielding an anomaly on the Weyl symmetry, an important gauge invariance. We shall prove this statement in this section and leave the problem for later treatment.

The starting point will be precisely the calculation of $\langle T^a_{\ a}\rangle$ in a 2d background close to flat space, knowing that this gives zero in the later one . Then

$$\delta\langle T^\alpha_{\ \alpha}\rangle = \frac{1}{4\pi}\int DXe^{-S}T^a_{\ a}(\sigma)\int d^2\sigma'\sqrt{g}\delta g^{bc}T_{bc}(\sigma') \tag{3.6.1}$$

If the change in the action is due to a Weyl transformation then $\delta g_{\mu\nu} = 2\lambda g_{\mu\nu}$ and as a consequence

$$\delta\langle T^\alpha_{\ \alpha}\rangle = -\frac{1}{2\pi}\int DXe^{-S}\int d^2\sigma'\sqrt{g}\lambda(\sigma')\, T^a_{\ a}(\sigma)\, T^b_{\ b}(\sigma') \tag{3.6.2}$$

We need now the OPE $T^a_{\ a}(\sigma)\, T^b_{\ b}(\sigma')$. We know the OPE in complex coordinates for TT, we should then apply a change of coordinates in the previous expression

$$T^a_{\ a}(\sigma)\, T^b_{\ b}(\sigma') = g^{a'b'}g^{c'd'}T_{a'b'}(z)\, T_{c'd'}(z') = 16T_{z\bar{z}}(z)\, T_{z'\bar{z}'}(z') \tag{3.6.3}$$

Now by the conservation of the stress energy tensor

$$\partial T_{z\bar{z}} + \bar{\partial} T_{zz} = 0 \tag{3.6.4}$$

This equation constitutes the bridge to calculate the OPE of $T_{z\bar{z}}T_{w\bar{w}}$ trough the OPE of $T_{zz}T_{ww}$

$$\partial_z T_{z\bar{z}} \partial_w T_{w\bar{w}} = \bar{\partial}_{\bar{z}} \bar{\partial}_{\bar{w}} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + ... \right) \tag{3.6.5}$$

The result is non zero. At least not everywhere due to the divergence at $z = w$. We saw earlier that $\partial\bar{\partial} \log |z|^2 = 2\pi\delta^{(2)}(z, \bar{z})$ which will be the reason for a non trivial result for $z = w$. Noting simply that

$$\frac{1}{(z-w)^4} = \frac{1}{6} \partial_z \partial_w^2 \frac{1}{z-w} \tag{3.6.6}$$

and since the other less divergent terms give zero we simply obtain

$$T_{z\bar{z}} T_{w\bar{w}} = \frac{c\pi}{6} \partial_z \bar{\partial}_{\bar{w}} \delta(z - w, \bar{z} - \bar{w}) \tag{3.6.7}$$

This OPE is zero everywhere except for $z = w$. The OPE in the other coordinates is obtained by 3.6.3

$$T^{\alpha}_{\ \alpha}(\sigma) T^{\mu}_{\ \mu}(\sigma') = \frac{8\pi}{3} c \partial_z \bar{\partial}_{\bar{w}} \delta(z - w, \bar{z} - \bar{w}) = -\frac{c\pi}{3} \partial^2 \delta(\sigma - \sigma') \tag{3.6.8}$$

which integrating by parts on 3.6.2 gives the result. It happens to be proportional to the Ricci Scalar (for small λ) since for this metric $R = -2e^{-2\lambda}\partial^2\lambda$

$$\langle T^{\alpha}_{\ \alpha} \rangle = \frac{c}{6} \partial^2 \lambda = -\frac{c}{12} R \tag{3.6.9}$$

For this expectation to be zero the only chance is to have $c = 0$ which does not seem very plausible taking into account the previous results. This will be solved later.

3.7. The state-operator map

There is an important result from conformal field theories that is the existence of a map between sates and local operators. At first there seems no reason for these two objects to be related at all: tipically in a quantum field theory we have local operators that live in a point in spacetime, while the states are specified at a fixed time but live over all the spatial dimensions.

In quantum mechanics we can write the propagator for a particle to move from a position i to a position f as:

$$G(x_i, x_f) = \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} Dx e^{iS} \tag{3.7.1}$$

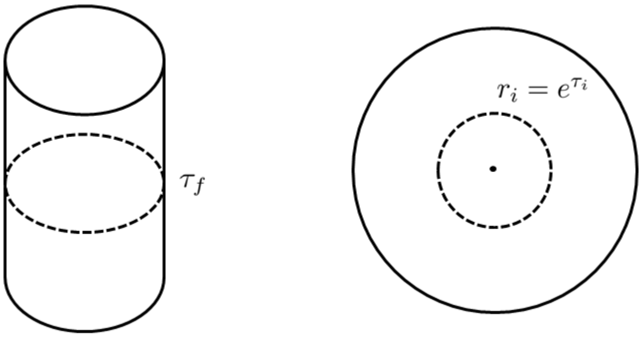


Figure 3.7.1.: Passage from the cylinder to the complex plane where the time quantization passes to radial quantization.

which enable us to write the final state up to a normalization factor

$$\psi_f\left(x_f,\tau_f\right)=\int dx_i G\left(x_i,x_f\right)\psi_i\left(x_i,\tau_i\right) \tag{3.7.2}$$

We now move to quantum field theory on the cylinder, making the natural passage to wave functionals

$$\psi_f\left[\phi_f\left(\sigma\right),\tau_f\right]=\int D\phi_i \int\limits_{\phi(\tau_i)=\phi_i}^{\phi(\tau_f)=\phi_f} D\phi e^{-S[\phi]}\psi_i\left[\phi_i\left(\sigma\right),\tau_i\right] \tag{3.7.3}$$

The passage to the plane is not a hard task, as we have saw, time evolution is now "radial evolution" governed by the dilaton operator, which simply means

$$\psi_f\left[\phi_f\left(\sigma\right),r_f\right]=\int D\phi_i \int\limits_{\phi(r_i)=\phi_i}^{\phi(r_f)=\phi_f} D\phi e^{-S[\phi]}\psi_i\left[\phi_i\left(\sigma\right),r_i\right] \tag{3.7.4}$$

The integration takes place on the ring limited inferiorly by $r_i = e^{\tau_i} = |z|$ as shown is figure 3.7.1.

If we now take the initial time to go to infinity past, then $r_i = 0$ and the integration is over all the disk rather than only in a ring. The initial state will only change the weight of the path integral at the origin. But in fact this is an integral of a local operator placed at $z = 0$. We conclude that each local operator $\mathcal{O}$ defines a different state of the theory:

$$\psi_f\left[\phi_f\left(\sigma\right),r_f\right]=\int^{\phi(r_f)=\phi_f} D\phi e^{-S[\phi]}\mathcal{O}\left(0,0\right) \tag{3.7.5}$$

This is strictly a property of conformal field theories since the key to this conclusion lies in the possibility of mapping the cylinder to the complex plane. These operators are known as the vertex operators associated with the state.

We now make some exploration on the free theory where it is possible to take the map more explicit. The mode expansion in complex coordinates w has the same form of the one derived for the ligh-cone coordinates

$$X^{\mu}(w,\bar{w}) = x^{\mu} - i\alpha' p^{\mu}\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\alpha^{\mu}_{m}e^{imw} + \tilde{\alpha}^{\mu}_{m}e^{-imw}\right)$$

The derivative of the field is simply $\partial_w X^{\mu}(w,\bar{w}) = -\sqrt{\frac{\alpha'}{2}}\sum\limits_{m}\alpha^{\mu}_{m}e^{imw}$. This is a primary operator of weights $(1,0)$ so we may pass easily from the cylinder to the plane (equation 3.4.4).

$$\partial X^{\mu} = -i\sqrt{\frac{\alpha'}{2}}\sum_{m}\frac{\alpha^{\mu}_{m}}{z^{m+1}} \tag{3.7.6}$$

Inverting the previous expression we obtain the modes

$$\alpha^{\mu}_{m} = i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi}z^{m}\partial X^{\mu} \tag{3.7.7}$$

The commutation relation among different modes can be obtained by the contour arguments of figure 3.5.1

$$[\alpha^{\mu}_{m},\alpha^{\nu}_{n}] = m\delta_{m+n}\eta^{\mu\nu} \tag{3.7.8}$$

To start the construction of the map we must start from some point, even though we may not know where it leads. It is natural to ask what happens if we insert the identity operator:$\psi[X_f] = \int^{X_f(r)} DXe^{-S[X]}$ Let us take the state and apply the modes 3.7.7: $\alpha^{\mu}_{m}\psi[X_f]$. It is not clear how an operator acts on the wave functional. What we should do is to consider the contour expression for α^{μ}_{m} and insert it in the path integral. We should consider $|z| < r$ and the field coming from the α^{μ}_{m} must also be integrated, this is

$$\alpha^{\mu}_{m}\psi[X_f] = i\sqrt{\frac{2}{\alpha'}}\int^{X_f(r)} DXe^{-S[X]}\oint\frac{dz}{2\pi}z^{m}\partial X^{\mu} \tag{3.7.9}$$

The action we are considering is the Polyakov action 3.0.4 which implies that if ∂X^{μ} is divergent at some point, the action will dominate, and the result will be zero. Consequently we should only sum over smooth ∂X^{μ}, but in this case the contour integral always gives zero for $m \geq 0$. This is precisely the usual definition of the vacuum state: it gives zero when acted by any annihilation operator. More precisely this is the vacuum state with zero momentum since it is annihilated also by α^{μ}_{0}.

Following this procedure of acting with the modes we can prove that the one mode excited states are given by

$$\alpha^{\mu}_{-m}|0\rangle = \int DXe^{-S[X]}\partial^{m}X(0) \tag{3.7.10}$$

We can proceed systematically to construct more complex state. The gravitons, for example, arise from the insertion of ∂ derivative and a $\bar{\partial}$ derivative.

We still do not know how to generate states with momentum different from zero. The OPE of ∂X^{μ} with $: e^{ipX(w)} :$ is even simpler to obtain then 3.4.19. This is enough to prove that the state

$$\int DXe^{-S[X]}e^{ipX(0,0)} \tag{3.7.11}$$

is the vacuum state with momentum p.

4. Path Integral Quantization

We are now in position to take a different quantization beside what was done so far, using path integral techniques. In this theory the conformal symmetry is a gauge invariance, and the usual way to deal with this in the path integral formulation is the well known Fadeev-Popov method. We expect then the appearance of ghosts to cancel the non physical degrees of freedom due to the gauge degrees, leaving us with the $D-2$ degrees of freedom predicted before with a non Lorentz invariant procedure like the light cone quantization. Beside being a more natural and elegant procedure, it is expected to get new insights and to solve, for example, the problem that emerged when we studied the energy-momentum tensor that suggested that we should obtain somehow $c=0$ (equation 3.6.9). We will show how can this statement make sense for a non-trivial theory.

4.1. From Fadeev-Popov method to ghosts

We have to fix gauge on the path integral

$$Z = \int DX\left(\sigma\right)Dg\left(\sigma\right)e^{-S_{Poly}[g,X]} \tag{4.1.1}$$

Due to the gauge invariance, all the metrics are physically equivalent. A general transformation on the metric is given by

$$g^{\alpha}_{ab} = e^{2\omega}\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}g_{cd} \tag{4.1.2}$$

The procedure follows by inserting in the path integral

$$1 = \Delta_{FP}\left[g\right]\int D\alpha\delta\left(g-g^{\alpha}\right) \tag{4.1.3}$$

The quantity $\Delta_{FP}\left[g\right]$ is the Fadeev-Popov determinant and it is not hard to see that it is gauge invariant. Following the Fadeev-Popov philosophy we drop the infinite multiplicative constant and arrive at

$$Z\left[g\right] = \int DX\left(\sigma\right)\Delta_{FP}\left[g\right]e^{-S_{Poly}[g,X]} \tag{4.1.4}$$

To obtain an expression for the Fadeev-Popov determinant we go back to its definition. Because of the Dirac delta we can consider only infinitesimal transformation (finite transformations are to far apart from the original metric g). Then[1]

$$\Delta_{FP}^{-1}[g] = \int D\xi D\omega\delta\left(2\omega g_{ab} + \xi_{a;b} + \xi_{b;a}\right) \tag{4.1.5}$$

It is useful to write the delta Dirac in a functional integral representation: $\delta\left(A_{ab}\right) = \int D\beta e^{2\pi i\int d^2\sigma\sqrt{g}\beta^{ab}A_{ab}}$. Putting this into the equation above will bring a trace term of β that has no derivatives. This means that, breaking $D\beta$ in integrals over symmetric, antisymmetric and trace, the last one is just a multiplicative factor that can be doped out. From now on we define β_{ab} to be symmetric and traceless (since the antisymmetric part gives zero). The Faddev-Popov determinant is written as

$$\Delta_{FP}^{-1}[g] = \int D\xi D\beta\exp\left(4\pi i\int d^2\sigma\sqrt{g}\beta^{ab}\nabla_b\xi_a\right) \tag{4.1.6}$$

This expression is simply the determinant of the inverse operator ∇. To invert $\Delta_{FP}^{-1}[g]$ we can pass to Grassman variables (or anticomuting fields) b_{ab} and c_a. This means that $\Delta_{FP}[g]$ will appear as an exponential of an integral on the worldsheet, resulting in a ghost action.

$$Z[g] = \int DXDbDce^{-S_{Poly}[g,X]-S_{ghost}[g,b,c]} \tag{4.1.7}$$

where

$$S_{\mathbf{ghost}} = \frac{1}{2\pi}\int d^2\sigma\sqrt{g}b_{ab}\nabla^a c^b \tag{4.1.8}$$

This is also a CFT. In the conformal gauge we have $\nabla^z = e^{-2\omega}\nabla_{\bar{z}}$, $\nabla^{\bar{z}} = e^{-2\omega}\nabla_z$ and $\sqrt{g} = e^{2\omega}$. This way we can write:

$$S_{\text{ghost}} = \frac{1}{2\pi}\int d^2z\left(b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}\right)$$

The action can be further simplified. We have $\nabla_z c^{\bar{z}} = \partial c^{\bar{z}} + \Gamma^{\bar{z}}_{az}c^a$ but $\Gamma^{\bar{z}}_{az} = \frac{1}{2}g^{z\bar{z}}\left(g_{az,\bar{z}} + g_{zz,\bar{z}} - g_{az,\bar{z}}\right)$ is allways zero. The same is true for $\nabla_{\bar{z}}c^z$ leading to

$$S_{\text{ghost}} = \frac{1}{2\pi}\int d^2z\left(b_{zz}\bar{\partial}c^z + b_{\bar{z}\bar{z}}\partial c^{\bar{z}}\right)$$

Because this holds for any metric of the conformal gauge, it means that the action is neutral under Weyl transformations. We write now $c^z = c$ and $c^{\bar{z}} = \bar{c}$ with analogous definitions for b. The equations of motion are very simple

$$\partial\bar{c} = \bar{\partial}c = \partial\bar{b} = \bar{\partial}b = 0 \tag{4.1.9}$$

showing b and c as holomorphic fields and $\bar{b}$ and $\bar{c}$ as anti-holomorphic fields.

The ghost energy-momentum tensor can be obtained in the usual way going back to the ghost action with general metric. The subtlety of this calculation is that the traceless property of b_{ab} is metric dependent, so we must add a term $b_{ab}g^{ab}$ as a Lagrange multiplier to account for this contribution

[1] The ; indicates covariant derivative.

and to guarantee that the trace is zero. This will result in

$$T = 2\left(\partial c\right) b + c\partial b,\ \bar{T} = 2\left(\bar{\partial}\bar{c}\right)\bar{b} + \bar{c}\bar{\partial}\bar{b} \tag{4.1.10}$$

4.2. Ghost and Conformal Field theory

We leave now the classical analysis to move to conformal field theory. The OPEs are derived in the same way as for X by acting inside the path integral with $\frac{\delta}{\delta b}$ and $\frac{\delta}{\delta c}$, giving

$$\bar{\partial} c\left(z,\bar{z}\right) b\left(w,\bar{w}\right) = \bar{\partial} b\left(z,\bar{z}\right) c\left(w,\bar{w}\right) = 2\pi\delta\left(z-w,\bar{z}-\bar{w}\right) \tag{4.2.1}$$

In an analogous way we integrate them provided that $\bar{\partial}\frac{1}{z} = 2\pi\delta\left(z,\bar{z}\right)$

$$b\left(z\right) c\left(w\right) = \frac{1}{z-w} + \ldots \tag{4.2.2}$$

The other OPEs have no singular parts. The energy-momentum tensor is the same with the normal ordering.

A natural calculation is the Tb and Tc OPEs. This is not hard to do explicitly by Wick contractions (keeping in mind their fermionic commutation relations). For $b\left(w\right)$

$$T\left(z\right) b\left(w\right) = 2 :\left(\partial c\right) b: b + :c\partial b: b = \frac{2b\left(z\right)}{\left(z-w\right)^2} - \frac{\partial b\left(z\right)}{z-w} + \ldots = \frac{2b\left(w\right)}{\left(z-w\right)^2} + \frac{\partial b\left(w\right)}{z-w} + \ldots \tag{4.2.3}$$

and for $c\left(w\right)$

$$T\left(z\right) c\left(w\right) = 2 :\left(\partial c\right) b: c + :c\partial b: c = \frac{2\partial c\left(z\right)}{z-w} - \frac{c\left(z\right)}{\left(z-w\right)^2} + \ldots = -\frac{c\left(w\right)}{\left(z-w\right)^2} + \frac{\partial c\left(w\right)}{z-w} + \ldots \tag{4.2.4}$$

Consequently we conclude that both c and b are primary fields with weights of -1 and 2 respectively.

Computing now the OPE of TT we can find the central charge of the ghosts

$$\begin{aligned} T\left(z\right) T\left(w\right) &= 4 :\left(\partial c\right) b::\left(\partial c\right) b: + 2 :\left(\partial c\right) b:: c\partial b: + 2 :c\partial b::\left(\partial c\right) b: + :c\partial b::c\partial b: \\ &= -13\frac{1}{(z-w)^4} - 4\frac{:b(z)c(w): + :c(z)b(w):}{(z-w)^3} - 4\frac{:b(z)\partial c(w): - :\partial c(z)b(w): + :c(z)\partial b(w): - :\partial b(z)c(w):}{(z-w)^2} + 2\frac{:\partial b(z)\partial c(w): + :\partial c(z)\partial b(w):}{z-w} \\ &= -\frac{13}{(z-w)^4} - \frac{4:b(w)\partial c(w): + 2:\partial b(w)c(w):}{(z-w)^2} + \frac{-:\partial^2 b(w)c(w): + 2:\partial^2 c(w)b(w): - 3:\partial b(w)\partial c(w):}{z-w} + \ldots \\ &= -\frac{13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \end{aligned} \tag{4.2.5}$$

This has the form one might expect for the OPE. Nonetheless, the most important fact is that the ghost central charge is $c = -26$. In this derivation we did not assumed as hypothesis the critical dimension, but indeed, this constitutes another proof of it (assuming one is working with free scalar fields). Only for $D = 26$ the central charges cancel each other and we get rid of the Weyl anomaly that was presented on section 3.6. We did not make any further assumption about the action besides conformal invariance. This means that any CFT with central charge $c = 26$ will work fine as well. Each different CFT describes a different background in which strings can propagate.

4.3. BRST Quantization

In section 2.2.1 we determined the spectrum of the string. This treatment is non covariant, adopts a particular gauge condition and considers only flat space. Then we introduced a covariant treatment, discovered “good” conditions for physical states but ignore the existence of ghost fields. It would be better if one could determine the spectrum for a general CFT, with a general gauge condition. This is known as BRST[2] quantization. We will start by working in the general method (not completely general, but quite enough) and then apply to strings.

Let us consider a general collection of path fields ϕ_i and a collection of gauge transformations that satisfy the algebra

$$[\delta_\alpha, \delta_\beta] = f^{\gamma}_{\ \alpha\beta}\delta_\gamma \tag{4.3.1}$$

The reason why we said that this is not completely general lies here in the first equation: we will not assume field dependence on f and we will not consider further terms on this algebra as terms proportional to equations of motion. We will apply Fadeev-Popov method, so we pick a gauge fixing

$$F\left[\phi\right] = 0 \tag{4.3.2}$$

and as usual we get the action

$$Z = \int D\phi DBDbDc e^{-S_1 - S_2 - S_3} \tag{4.3.3}$$

where S_1 is the original action and

$$S_2 = -iBF\left[\phi\right], \ S_3 = bc^\alpha \delta_\alpha F\left[\phi\right] \tag{4.3.4}$$

are respectively due to the delta functional and the determinant functional. With this total action, it would be useful to find a symmetry, completing the gauge symmetry of the fields ϕ_i. This symmetry

[2]Due to Becchi-Rouet-Stora-Tyutin

exists and it is know as BRST transformation. Explicitly

$$\begin{aligned}\Delta\phi_i &= -i\lambda c^\alpha \delta_\alpha \phi_i \\ \Delta B &= 0 \\ \Delta b &= \lambda B \\ \Delta c^\alpha &= \tfrac{i}{2}\lambda f^\alpha_{\ \mu\nu} c^\mu c^\nu\end{aligned} \tag{4.3.5}$$

The λ parameter must be anticommuting for the previous expression to make sense. Obviously S_1 remains unchanged. S_2 only changes due to F and its variation is $-B\lambda c^\alpha \delta_\alpha F$. The variations of S_3 is

$$\lambda B c^\alpha \delta_\alpha F + \frac{i}{2} b\lambda f^\alpha_{\ \mu\nu} c^\mu c^\nu \delta_\alpha F - ibc^\alpha \delta_\alpha \lambda c^\beta \delta_\beta F$$

The first term is canceled by the change in S_3. It is not hard to see that the other two cancel by using the commutation relation 4.3.1 and in particular the anti-symmetry of the lower indices of f. So this is in fact a symmetry. This transformation also has the property

$$\Delta (bF) = i\lambda (S_2 + S_3) \tag{4.3.6}$$

What we want to study is what happens when we change the condition for the gauge fixing 4.3.2. For that we consider a small change δF and see its effect on the scalar product $\langle f | \, i \rangle$ for given configurations. A different choice for the gauge fixing cannot alter the physical system, so, for physical states, this scalar product may not vary. The action is only affected in S_2 and S_3. Using the path integral representation of $\langle f | \, i \rangle$ we expand in Taylor series the variation of the action and use the previous expression to obtain

$$\lambda\delta \langle f | \, i \rangle = i\delta \langle f | \, \Delta (b\delta F) \, | i \rangle$$

We can write the change on the fields $b\delta F$ as the anticomutator with the conserved charge, giving

$$-\lambda\delta \{Q, b\delta F\} \, | i \rangle = 0 \tag{4.3.7}$$

Therefore, the condition for physical states is

$$Q \, | i \rangle = \langle f | \, Q = 0$$

We will assume that the charge is real, otherwise another symmetry should emerge. The condition is simply

$$Q \, | \psi \rangle = 0 \tag{4.3.8}$$

The charge is nilpotent. To see this we use the fact that because of Q conservation it must commute with the change on the Hamiltonian which is to say $[Q, \{Q, b\delta F\}] = 0$. This result in $[Q^2, b\delta F] = 0$

implying $Q^2 = const$. To see that this constant is indeed zero is just a matter of computation applying the charge twice on the fields.

$$Q^2 = 0 \tag{4.3.9}$$

The nilpotence has a particular consequence, it means that for any given state $|\chi\rangle$ the state $Q|\chi\rangle$ is physical. Furthermore they are orthogonal to all physical states (including itself), therefore are called null states. Physical states that differ from each other by a null state show the same inner product with all physical states, which implies they are physically equivalent. As a consequence physical states should be defined to live within an equivalent class. In general, states that are annihilated by Q are called closed, while states of the form $Q|\chi\rangle$ are called exact.

4.3.1. BRST quantization for 2d CFT

We will now apply the above procedure to string theory. Let us clarify the action for the present case.

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{i}{4\pi}\int d^2\sigma\sqrt{g}B^{ab}\left(h_{ab} - g_{ab}\right) + \frac{1}{2\pi}\int d^2\sigma\sqrt{g}b_{ab}\nabla^a c^b \tag{4.3.10}$$

This actions are defined to be respectively S_m, S_B and S_{bc}. The m of the first one stands for "matter" and it emphasizes that we could consider other CFTs.

Consider an infinitesimal conformal transformation. The action S_B is needed to make sense of the BRST transformation here. The fields X^μ only change because of the diffeomorphism transformation $\delta\sigma^a = \xi^a(\sigma)$. The first step is to choose a basis for this transformation. We pick the delta Dirac basis where $\delta\sigma^a = \int d^2x\varepsilon^a(x)\,\delta(x-\sigma)$. The fields change as $\delta X^\mu = -\xi^a(\sigma)\,\partial_a X^\mu$, or in the delta basis $\delta X^\mu = -\int d^2x\xi^a(x)\,\delta(x-\sigma)\,\partial_a X^\mu$ which means we found our transformation operators

$$\delta_{ax} = -\delta(x-\sigma)\,\partial_a \tag{4.3.11}$$

The next step is to find its algebra. We have now two indices: a and x. Obviously the sum turns into an integral for x.

$$\begin{aligned} [\delta_{ax}, \delta_{by}] &= \textstyle\int d^2z f^{cz}_{xa\ yb}\delta_{cz} = -\left(\delta(x-\sigma)\,\partial_a\delta(y-\sigma)\,\partial_b - \delta(y-\sigma)\,\partial_b\delta(x-\sigma)\,\partial_a\right) \\ &= -\textstyle\int d^2z\left(\delta^c{}_b\delta(x-z)\,\frac{\partial}{\partial z^a}\delta(y-z) - \delta^c{}_a\delta(y-z)\,\frac{\partial}{\partial z^b}\delta(x-z)\right)\delta(z-\sigma)\,\partial_c \end{aligned} \tag{4.3.12}$$

Consequently the structure constants are

$$f^{cz}_{xa\ yb} = \delta^c{}_b\delta(x-z)\,\frac{\partial}{\partial z^a}\delta(y-z) - \delta^c{}_a\delta(y-z)\,\frac{\partial}{\partial z^b}\delta(x-z) \tag{4.3.13}$$

Now that we have identified the algebra of the transformations 4.3.1 we have to write how the fields transform in a BRST transformation 4.3.5.

$$\Delta X^{\mu} = -i\lambda \int d^2 x c^a (x) \delta_{ax} X^{\mu} = i\lambda c^a (\sigma) \partial_a X^{\mu} \tag{4.3.14}$$

The transformations of b and B are simple and for c:

$$\Delta c^c = \tfrac{i}{2}\lambda \int d^2 x d^2 y f^{c\sigma}_{ax\ by} c^a (x) c^b (y) = \tfrac{i}{2}\lambda \left(\int d^2 y \tfrac{\partial}{\partial \sigma^a} \delta (y-\sigma) c^a (\sigma) c^c (y) - \int d^2 x \tfrac{\partial}{\partial \sigma^b} \delta (x-\sigma) c^c (x) c^b (\sigma) \right)$$
$$= i\lambda \partial_a c^a (\sigma) c^c (\sigma) \tag{4.3.15}$$

What we will do now is to obtain the transformation of b_{ab} trough BRST writing B as a function of other fields. This is possible trough the equation of motion of the metric, that it is

$$-\frac{\sqrt{g}}{4\pi} \left(T^m_{ab} + T^g_{ab} \right) - \frac{i}{4\pi} \sqrt{g} B_{ab} - \frac{i}{8\pi} \sqrt{g} B_{cd} \left(h^{cd} - g^{cd} \right) g_{ab} = 0 \tag{4.3.16}$$

resulting in

$$B_{ab} = i \left(T^m_{ab} + T^g_{ab} \right) \tag{4.3.17}$$

so b transforms as

$$\Delta b_{ab} = i\lambda \left(T^m_{ab} + T^g_{ab} \right) \tag{4.3.18}$$

The conserved currents are determined by Noether method. There is a shortcut for this that we already used when we obtained the conserved currents in equation 3.3.19. We obtained the variation of the action for a fixed metric knowing that without it fixed, the action would be conformal invariant. We obtain the current in the same way with $v(z) = c(z)$. The result is

$$j = cT^m + \frac{1}{2} : cT^g :$$

There is a slight problem that we would like to avoid: j as it is defined is not a tensor. In fact T does not transform like a tensor, but rather as in 3.4.25. Adding a total derivative will not alter the BRST charge, picking the correct term will give a tensor. We can see by direct computation that if we add $\frac{3}{2}\partial^2 c$ the result is a tensor, so, from now on, we consider

$$j = cT^m + : bc\partial c : + \frac{3}{2} \partial^2 c \tag{4.3.19}$$

The ghost tensor is now explicit, with analogous result for $\bar{j}$. The BRST charge is

$$Q = \frac{1}{2\pi i} \oint \left(dz j - d\bar{z} \bar{j} \right) \tag{4.3.20}$$

We consider the expansion of the ghost modes on an open string[3]. Since they are (anti)holomorphic

$$b(z) = \sum_m \frac{b_m}{z^{m-1}}$$
$$c(z) = \sum_m \frac{c_m}{z^{m-1}} \tag{4.3.21}$$

Furthermore we can follow the procedure used before to find their anticomutation relations trough classical poison brackets, or through contour arguments. The result is

$$\{b, c\} = 1$$
$$\{b_m, c_n\} = \delta_{m+n} \tag{4.3.22}$$

With this we can compute explicitly the BRST charge. Picking up the residues gives

$$Q = \sum_n c_{-n} L_n^m + \sum_{m,n} \frac{m-n}{2} : c_m c_n b_{m-n} : + a^B c_0 \tag{4.3.23}$$

The last term is there because of the particular choice of order. Again by contour arguments pictured on figure 3.5.1 we get

$$\{Q, b_n\} = L_n^m + L_n^g \tag{4.3.24}$$

Particularizing for $n = 0$ it is a matter of computation to see that $a^B = -1$.

4.3.2. The Spectrum

We set for all modes $(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu$ including the same kind of relation for ghosts. We resume here the commutation relations

$$[x^\mu, p^\nu] = -i\eta^{\mu\nu},\ [\alpha_m^\mu, \alpha_n^\nu] = -m\delta_{m+n}\eta^{\mu\nu},\ \{b_m, c_n\} = \delta_{m+n} \tag{4.3.25}$$

The vacuum state is defined to be annihilated by all the positive modes

$$\alpha_m^\mu |0\rangle = b_m |0\rangle = c_m |0\rangle = 0 \tag{4.3.26}$$

As in the case of the matter part, where we still have to label the state with a momentum k due to the zero mode α_0^μ, we need an extra label for the the zero mode ghosts that commute with the Hamiltonian. There are only two possibilities as is explained on figure 4.3.1.

Let us denote this states by

$$c_0 |\uparrow\rangle = b_0 |\downarrow\rangle = 0,\ b_0 |\uparrow\rangle = |\downarrow\rangle,\ c_0 |\downarrow\rangle = |\uparrow\rangle \tag{4.3.27}$$

[3]Closed strings follow the same process but with heaviest notation.

$$\begin{tikzcd}
& \left|\phi\right\rangle \arrow[dl, "c_0"'] \arrow[dr, shift right, "b_0"'] & \\
0 & & b_0\left|\phi\right\rangle \arrow[ul, shift right, "c_0"'] \arrow[ll, "b_0"]
\end{tikzcd}$$

Figure 4.3.1.: We start with a state that is annihilated by c and call it $\left|\phi\right\rangle$. Then this sates is also annihilated by bc and by the anticomutation relation $b\left|\phi\right\rangle$ is eigenstate of c with eigenvalue 1. Noting that b^2 is zero we have the diagram.

Now we glue this to the Fock space generated by the leader operators α, and work on a general state $P(\alpha)\left|k,\updownarrow\right\rangle$ where k represents the ground state of α's with momentum k, $\updownarrow$ represents the two possibilities for the ghosts and $P(\alpha)$ is acting on this ground states as a polonium of the α's.

Let us consider the ground states and apply the BRST charge 4.3.23. We easily see that for every term on the charge there is at least one lowering operator or zero modes. As a consequence the possible pathological terms, in the sense that it may give non zero, are $c_0(L_0-1)\left|k,\updownarrow\right\rangle$. We conclude

$$Q\left|k,\updownarrow\right\rangle = 0 \text{ for} \begin{cases} \text{any } k \text{ and } \uparrow \\ \\ k^2 = \frac{1}{\alpha'} \text{ and } \downarrow \end{cases} \tag{4.3.28}$$

The ground states with $\uparrow$ are automatically on shell. It is therefore more natural to consider only the $\downarrow$ states using the subsidiary condition for physical states

$$b_0\left|\psi\right\rangle = 0 \tag{4.3.29}$$

Because of relation 4.3.24 this later condition also implies

$$L_0\left|\psi\right\rangle = 0 \tag{4.3.30}$$

This give us the mass shell condition. The L_0 operator is

$$L_0 = p^\mu p_\mu + \sum_{n\geq 1} n\left(N_{bn} + N_{cn} + \sum_\mu N_{\mu n}\right) - 1 \tag{4.3.31}$$

From now the ground states are denoted just by $\left|k\right\rangle$ assuming that the previous conditions are fulfilled (equivalently $\updownarrow=\downarrow$).

We will be forced to take a new definition of the scalar product. Indeed, if we use the usual rule, all the ground states are all orthogonal to each other:

$$\left\langle k' \mid k\right\rangle = \left\langle k'\right| b_0c_0 + c_0b_0\left|k\right\rangle = 0 \tag{4.3.32}$$

Even for $k=k'$ this is zero because b_0 annihilates both the ket and the bra. Even though it is possible to construct an adequate scalar product, we will not need it and will just refer to is as $\langle \; \| \; \rangle$.

For each k the most general state one can construct is of the form

$$|\psi_1\rangle = (e_m \cdot \alpha_{-1} + e_b b_{-1} + e_c c_{-1})\,|k\rangle \tag{4.3.33}$$

e_m is a vector of 26 components while e_b and e_c are scalars. This means we have 28 states at this level. The norm of this states is

$$\langle \psi_1 \;\|\; \psi_1 \rangle = \left(e^m \cdot e^{m*} + e^{c*} e^b + e^{b*} e^c\right) \langle k \;\|\; k \rangle \tag{4.3.34}$$

Consequently, if we adopt an orthonormal basis there are 26 states of positive norm and 2 with negative one.

Now we apply the BRST charge to $|\psi_1\rangle$. Some computation leads directly to

$$\sqrt{2\alpha'}\left(k \cdot e^m c_{-1} + e^b k \cdot \alpha_{-1}\right)|\psi_1\rangle \tag{4.3.35}$$

The conditions for physical states of the first level is

$$k \cdot e^m = e^b = 0 \tag{4.3.36}$$

These conditions reduce the number of states to 24 states of positive norm and 2 with null norm. These ones are the exact sates and we stay with 24 vector states, with no mass and positive norm. This is a general result. At each level we have two extra positive norm sates and two negative norm sates. The first two are gone when we consider only closed states, and the negative norm become null and are discarded for being exact states, or formally, by the class of equivalence relations.

5. Strings on Background Fields

Until now all the discussion was made considering strings that propagate in flat space. In this section we consider the case of a non-trivial background, i,e., not flat. The action is then

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu}(X) \tag{5.0.1}$$

We can no longer reduce the theory to a free one by a suitable choice of gauge, i.e., the conformal gauge. Instead we will get always an interacting theory. Before proceeding it is important to understand what are we doing. Putting the theory on a curved background we are intuitively coupling the string with gravity but the graviton already appeared on the string quantization. This background should then be built from gravitons. We can see this by expanding the metric as a perturbation to flat space separating clearly the action due to interaction

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X) \longrightarrow S_{int} = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu\partial_b X^\nu h_{\mu\nu}(X) \tag{5.0.2}$$

We see that the interaction term is precisely the vertex operator for the graviton: recall the results from section 3.7 and note $h_{\mu\nu}$ is symmetric and traceless. Treating this as a perturbation in the path integral

$$\int DXDge^{-S} = \int DXDge^{-S}\left(1 - S_{int} + \frac{1}{2}S_{int}^2 - ...\right) \tag{5.0.3}$$

we also see that corresponds to a coherent state of gravitons that will modify the flat space metric.

Back to the picture of an interacting theory we can start by expanding X^μ

$$X^\mu(\sigma) = x^\mu + \sqrt{\alpha'}Y^\mu(\sigma) \tag{5.0.4}$$

Expanding in the adimensional fluctuations Y^μ

$$G_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu = \alpha'\left(G_{\mu\nu}(x) + \sqrt{\alpha'}G_{\mu\nu,\alpha}(x)Y^\alpha + \frac{\alpha'}{2}G_{\mu\nu,\alpha\beta}(x)Y^\alpha Y^\beta + ...\right)\partial_a Y^\mu\partial_b Y^\nu \tag{5.0.5}$$

This can be regarded as a theory with infinite coupling constants with interacting fields Y^μ. Roughly, this expansion is expected to be valid when the ratio $\frac{\sqrt{\alpha'}}{r_c}$ is small ($\frac{\partial G}{\partial X} \sim \frac{1}{r_c}$). This new theory defined in 5.0.1 is conformally invariant classically, but after regularization the physics tipically depends on the scale trough the UV cut-off μ. As we just saw, our interacting theory has an infinite number of couplings all suitably condensed in $G_{\mu\nu}(X)$. The β-function is then a β-functional.

$$\beta_{\mu\nu}(G) \sim \frac{\partial G_{\mu\nu}}{\partial \log \mu} \tag{5.0.6}$$

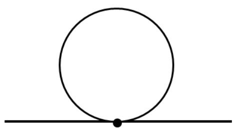
To preserve the scale invariance we have to guarantee

$$\beta_{\mu\nu}(G) = 0 \tag{5.0.7}$$

This will obviously constrain the background metric $G_{\mu\nu}$. To obtain this condition in the present theory we move to the Riemann normal coordinates where the action will be (up to second order)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_a Y^\mu \partial^a Y^\nu \left(\eta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\alpha\nu\beta} Y^\alpha Y^\beta \right) \tag{5.0.8}$$

We will now proceed with dimensional regularization. The divergence in the theory comes from the one loop-diagram



$$\left\langle Y^\alpha(\sigma) Y^\beta(\sigma') \right\rangle = 2\pi\eta^{\alpha\beta} \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik\cdot(\sigma-\sigma')}}{k^2} \tag{5.0.9}$$

Figure 5.0.1.: One loop diagram responsible for the divergence.

In this procedure we consider

$$\left\langle Y^\alpha(\sigma) Y^\beta(\sigma') \right\rangle \longrightarrow 2\pi\eta^{\alpha\beta} \int \frac{d^{2+\varepsilon}k}{(2\pi)^{2+\varepsilon}} \frac{e^{ik\cdot(\sigma-\sigma')}}{k^2} \sim \frac{\eta^{\alpha\beta}}{\varepsilon}, \ \sigma \to \sigma' \tag{5.0.10}$$

So we must subtract the following term to the action in order to obtain a finite theory

$$\Delta S = -\frac{1}{12\pi\varepsilon} \int d^2\sigma \partial_a Y^\mu \partial^a Y^\nu R_{\mu\nu} \tag{5.0.11}$$

We have then

$$R_{\mu\alpha\nu\beta} Y^\alpha Y^\beta \partial_a Y^\mu \partial^a Y^\nu \to R_{\mu\alpha\nu\beta} Y^\alpha Y^\beta \partial_a Y^\mu \partial^a Y^\nu - \frac{1}{\epsilon} \partial_a Y^\mu \partial^a Y^\nu R_{\mu\nu} \tag{5.0.12}$$

Requiring the Weyl invariance, at this order, is then equivalent to impose that the background must obey the Einstein equations in the vacuum

$$R_{\mu\nu} = 0 \tag{5.0.13}$$

5.1. Other modes of the string: B field and Dilaton

It was natural to study a string on a non-trivial background and, as we saw, it was directly associated with the graviton massless state. The next step is to introduce the other two massless states of the string, an anty-symmetric tensor field $B_{\mu\nu}$ and a scalar field ϕ (dilaton). The total action will be $S = S_1 + S_2 + S_3$ where S_1 is the action 5.0.1, S_2 is the B field action and S_3 the dilaton action. S_2 has an obvious form, and, furthermore, we can take the argument that lead us to conclude that the background field $G_{\mu\nu}$ is generated by gravitons to find the new term of the action trough the vertex operator of the B field. This is

$$S_2 = \frac{1}{4\pi\alpha'} \int d^2\sigma \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \tag{5.1.1}$$

where ϵ^{ab} is the antisymmetric tensor density of weight minus one.

The inclusion of the dilaton is not so obvious, and can be archived by

$$S_3 = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \phi(X) R^{(2)} \tag{5.1.2}$$

where $R^{(2)}$ is the worldsheet Ricci scalar. We can see this by considering the two dimensional Einstein-Hilbert action:

$$\chi = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} R^{(2)} \tag{5.1.3}$$

Its variation with respect to the metric is, obviously, well known giving in the integrand $R^{(2)}_{ab} - \frac{1}{2} h_{ab} R^{(2)} = 0$, but this is an identity on two dimension which means that χ is a topological invariant (in fact it is known as the Euler characteristic of the two-dimensional manifold). This way χ does not contribute to the dynamic of the string but gives us a way to incorporate the dilaton as in 5.1.2. The inclusion of such a term has one direct implication: the breakdown of Weyl invariance at a classical level. This term will be seen as a higher loop order and will be used to cancel the Weyl anomaly coming from the other two.

We now write the two-dimensional fields X^μas the sum of "classical" X_0^μ (that obey the classical equations of motion by construction) plus a "quantum" field x^μ taking the rule of variable of integration on the path integral.

$$X^\mu(\sigma) = X_0^\mu(\sigma) + x^\mu(\sigma) \tag{5.1.4}$$

Expanding the action on x^μ leads to a well defined perturbation theory where we can derive the Feynman rules. Nonetheless this procedure is not manifestly covariant since x^μ does not transform like a vector (it is the difference of two coordinates). This can be done by changing to normal coordinates where the coordinates are the components of the vector tangent to the geodesic that connects the points X_0^μ and $X_0^\mu + x^\mu$, denoted by t^μ. The relevant expressions regarding this transformations are

$$x^\mu = t^\mu - \tfrac{1}{2}\Gamma^\mu_{\sigma_1\sigma_2} t^{\sigma_1} t^{\sigma_2} + ..., \text{ Transformation of coordinates}$$

$$\partial_a x^\mu = \partial_a t^\mu + \Gamma^\mu_{\sigma_1\sigma_2}\partial_a X_0^{\sigma_1} t^{\sigma_2} + \tfrac{1}{3} R^\mu{}_{\alpha\beta\nu}\partial_a X_0^\nu t^\alpha t^\beta + ..., \text{ Expantion of the derivative}$$

$$G_{\mu\nu}(X_0 + x) = G_{\mu\nu}(X_0) + \tfrac{1}{3} R_{\mu\alpha\beta\nu}(X_0)\, t^\alpha t^\beta + ..., \text{ Expantion of the background metric}$$

We will denote for simplicity $\nabla_a t^\mu = \partial_a t^\mu + \Gamma^\mu_{\sigma_1\sigma_2}\partial_a X_0^{\sigma_1} t^{\sigma_2}$. In this coordinates the action S_1 is, up to the leading order in t^μ

$$\begin{gathered} S_1[X_0 + x] = S_1[X_0] + \tfrac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{g} g^{ab}\partial_a X_0^\mu \nabla_b t^\nu G_{\mu\nu}(X_0) + \\ + \tfrac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g} g^{ab}\left(\nabla_a t^\mu \nabla_b t^\nu G_{\mu\nu}(X_0) + R_{\mu\alpha\beta\nu}\partial_a X_0^\mu \partial_b X_0^\nu t^\alpha t^\beta\right) \\ + \tfrac{1}{12\pi\alpha'}\int d^2\sigma\sqrt{g} g^{ab}\left(4R_{\mu\alpha\beta\nu}\partial_a X_0^\mu \nabla_b t^\nu t^\alpha t^\beta + R_{\mu\alpha\beta\nu}\nabla_a t^\mu \nabla_b t^\nu t^\alpha t^\beta\right) \end{gathered} \tag{5.1.5}$$

The term linear in t^μ, the first integral, can be integrated by parts to give zero providing that X_0^μ obey the classical equations of motion. In the first integral of the second line we have the kinetic term of the theory (quadratic term on the two derivatives of the field) that will give us the propagator. Due to the coupling to the metric $G_{\mu\nu}(X_0)$ the propagator will be non-trivial. Because of this we introduce the Vielbein formalism

$$t^i = e^i_\mu(X_0)\, t^\mu,\ e^i_\mu(X_0)\, e^j_\nu(X_0)\,\eta_{ij} = G_{\mu\nu}(X_0) \tag{5.1.6}$$

which allow us to write this term in a simplest way

$$\nabla_a t^\mu \nabla_b t^\nu G_{\mu\nu}(X_0) = (\nabla_a t)^i (\nabla_b t)^i \tag{5.1.7}$$

where $(\nabla_a t)^i = \partial_a t^i + \omega^{ij}_\mu \partial_a X_0^\mu t^j$ with ω^{ij}_μ being the spin connection. Neither one of this changes of variables changes the path integral measure.

We proceed now to S_2 in a similar way. One obtains

$$\begin{gathered} S_2[X_0 + x] = S_2[X_0] + \tfrac{1}{2\pi\alpha'}\int d^2\sigma\epsilon^{ab}\left(\partial_a X_0^\mu \nabla_b t^\nu B_{\mu\nu}(X_0) + \tfrac{1}{2}\nabla_\lambda B_{\mu\nu}\partial_a X_0^\mu \partial_b X_0^\nu t^\lambda\right) + \\ + \tfrac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{ab}\left(\nabla_a t^\mu \nabla_b t^\nu B_{\mu\nu}(X_0) + 2\nabla_\lambda B_{\mu\nu}\partial_a X_0^\mu \nabla_b t^\nu t^\lambda\right) \\ + \tfrac{1}{8\pi\alpha'}\int d^2\sigma\epsilon^{ab}\left(\nabla_\lambda\nabla_\sigma B_{\mu\nu} + R^\rho{}_{\lambda\sigma\mu} B_{\mu\nu}(X_0)\right)\partial_a X_0^\mu \partial_b X_0^\nu t^\lambda t^\sigma + ... \end{gathered} \tag{5.1.8}$$

Again the first line of linear terms on t^μ drops out by virtue of the classical equations of motion. By other side, the action of the B field 5.1.1 is invariant under the gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\mu \Lambda_\nu$$

where $\Lambda_\mu(X)$ is some vector function. Because of this gauge invariance, it is useful to write the covariant derivative terms of the B field as a function of the tensor field strength:

$$H_{\mu\nu\sigma} = \nabla_\mu B_{\nu\sigma} + \nabla_\sigma B_{\mu\nu} + \nabla_\nu B_{\sigma\mu} \tag{5.1.9}$$

This field strength is clearly gauge invariant and, in fact, as expected, the physics will only be dependent of it. With some integration by parts the quadratic terms of the action, second and third line, gives

$$\frac{1}{4\pi\alpha'} \int d^2\sigma \epsilon^{ab} \left(H_{\mu ij}(X_0)\, \partial_a X_0^\mu \nabla_b t^i t^j + \frac{1}{2} \nabla_\lambda H_{\mu\nu j} \partial_a X_0^\mu \partial_b X_0^\nu t^i t^j \right) \tag{5.1.10}$$

In an analog way we obtain the Dilaton action

$$S_3[X_0 + x] = S_3[X_0] + \frac{1}{4\pi} \int d^2\sigma \sqrt{g} R^{(2)} \nabla_i \phi(X_0)\, \eta^i + \frac{1}{8\pi} \int d^2\sigma \sqrt{g} R^{(2)} \nabla_i \nabla_j \phi(X_0)\, \eta^i \eta^j \tag{5.1.11}$$

We are now in position in proceeding with perturbation theory. Again for this procedure to be valid one must assume that the fields are small varying when compared with characteristic length $\sqrt{\alpha'}$.

We are looking for the Weyl anomaly, that is, the variation of the effective action with respect to a scale factor of the worldsheet metric. We will adopt an alternative procedure rather then a direct computation which will allow us to save some time. We achieve this by considering the energy momentum-tensor conservation like in 3.6.4 that given T_{zz} will be the shortcut to obtain a non-zero $T_{z\bar{z}}$. Because we are considering the conservation of energy-momentum tensor we clearly keep the reparametrization invariance leading us to the break of the Weyl invariance. Not requiring this conservation condition, that is, giving up of the reparametrization invariance, would able us to keep the Weyl invariance, but still, the price is higher that way.

The conservation of energy-momentum tensor in momentum space reads (we enphasize the mean value here)

$$p_z \left\langle \tilde{T}_{z\bar{z}} \right\rangle + p_{\bar{z}} \left\langle \tilde{T}_{zz} \right\rangle = 0 \tag{5.1.12}$$

It is legitime to obtain $T_{z\bar{z}}$ from this equation since T_{zz} is finite. The expansion goes as follows

$$\tilde{T}_{zz}(p) = \int \frac{d^2 z}{2\pi} T_{zz}(z)\, e^{-ip\cdot z} = -\frac{1}{\alpha'} \int \frac{d^2 z}{2\pi} \left(2 : \partial X_0^\mu \partial x^\nu : + : \partial x^\mu \partial x^\nu : \right) G_{\mu\nu} e^{-ip\cdot z} \tag{5.1.13}$$

because the tensor is zero classically. Now we change the coordinates $x^\mu \to t^i$. The term of order one in the integral drops out because it vanishes in the path integral. Higher orders in the curvatures will not be considered. This is more explicitly justified later, for that purpose for now we write schematically $\partial x^\nu \sim t^i + \mathcal{O}(R)$ as one can easily verify. On the other hand we should write exactly the second one (up to the leading order) where we should make use of the inverted expression of 5.1.6:

$t^{\mu} = G^{\mu\nu} e^{j}_{\nu} \eta_{ij} t^{i}$. Then

$$\partial x^{\mu} \partial x^{\nu} = \left(G^{\mu\lambda} e^{l}_{\lambda} \eta_{il} \partial t^{i} + \Gamma^{\mu}_{\sigma_1 \sigma_2} \partial X_0^{\sigma_1} G^{\sigma_2 \lambda} e^{j}_{\lambda} \eta_{ij} t^{i} + \mathcal{O}\left(R\right) \right) \\ \times \left(G^{\nu\sigma} e^{k}_{\sigma} \eta_{jk} \partial t^{j} + \Gamma^{\nu}_{\sigma_1 \sigma_2} \partial X_0^{\sigma_1} G^{\sigma_2 \lambda} e^{j}_{\lambda} \eta_{ij} t^{i} + \mathcal{O}\left(R\right) \right) G_{\mu\nu}$$

For now we focus on the S_1 piece. Looking at 5.1.5 we ignore the first line (as we discussed the integral is zero and $S\left[X_0\right]$ factorizes) and the third because is of higher order. We have then the second line that will bring

$$e^{\frac{1}{8\pi\alpha'} \int d^2 z g^{ab} \partial_a t^i \partial_b t^i} \left(1 + \frac{1}{8\pi\alpha'} \int d^2 z g^{ab} \omega^{ij}_{\mu} \partial_b X_0^{\mu} \partial_a t^i t^j - \frac{1}{8\pi\alpha'} \int d^2 z \frac{1}{D} R_{\mu\nu} \eta_{ij} g^{\hat{a}b} \partial_a X_0^{\mu} \partial_b X_0^{\nu} t^i t^j + \mathcal{O}\left(\omega^2\right) \right) \tag{5.1.14}$$

The term involving the connections cannot give a covariant results (one would need a derivative to form a curvature tensor) and then their combination must give zero. This way we forget about them and we arrive with just one contribution

$$\frac{\frac{1}{\alpha'} \int Dt e^{\frac{1}{8\pi\alpha'} \int d^2 z g^{ab} \partial_a t^i \partial_b t^i} \int \frac{d^2 z}{2\pi} \partial t^i \partial t^j \eta_{ij} e^{-ip\cdot z} \frac{1}{8\pi\alpha'} \int d^2 z' \frac{1}{D} R_{\mu\nu} \eta_{kl} g^{ab} \partial_a X_0^{\mu} \partial_b X_0^{\nu} t^k t^l}{\int Dt e^{\frac{1}{4\pi\alpha'} \int d^2 \sigma \eta^{ab} \partial_a t^i \partial_b t^i}} \tag{5.1.15}$$

or switching for the momentum space

$$\frac{1}{8\pi\alpha'^2} \frac{1}{\int D\tilde{t} e^{\frac{1}{8\pi\alpha'} \int d^2 p p^2 \left|\tilde{t}^i(p)\right|^2}} \int D\tilde{t} \left(e^{\frac{1}{8\pi\alpha'} \int d^2 p p^2 \left|\tilde{t}^i(p)\right|^2} \frac{1}{D} \int \frac{d^2 p d^2 q d^2 l}{(2\pi)^2} p_z \left(p_z + k_z\right) \tilde{t}^i\left(p\right) \left(\tilde{t}^j\left(p+k\right)\right)^* \tilde{t}^k\left(q\right) \tilde{t}^l\left(l\right) \eta_{ij} \eta_{kl} \\ \times \mathcal{FT}\left(R_{\mu\nu} g^{ab} \partial_a X_0^{\mu} \partial_b X_0^{\nu} \right) \left(q + l\right) \right) \tag{5.1.16}$$

$\mathcal{FT}$ stands for Fourier transform. This reduces to a simple path integral that can be evaluated breaking the fields $\tilde{t}$ in real and imaginary part. The final result is

$$\left\langle \tilde{T}_{zz}\left(k\right) \right\rangle = \frac{1}{2\pi} \mathcal{TF}\left(R_{\mu\nu} g^{ab} \partial_a X_0^{\mu} \partial_b X_0^{\nu} \right)\left(k\right) \int d^2 p \frac{p_z\left(p_z + k_z\right)}{p^2 \left(p+k\right)^2} \tag{5.1.17}$$

The integral gives $-\frac{\pi}{2}\frac{k_z}{k_{\bar{z}}}$. From 5.1.12 is clear that

$$\left\langle T_{z\bar{z}}\left(z\right) \right\rangle = \frac{1}{4} R_{\mu\nu} g^{ab} \partial_a X_0^{\mu} \partial_b X_0^{\nu} \tag{5.1.18}$$

This is not the full anomaly, we also have to count the contributions of S_2 and S_3. From S_2 the situation is quite analogous to this previous treatment. We will have to consider two diagrams that will give the results for $\left\langle \tilde{T}_{z\bar{z}}\left(z\right) \right\rangle$

$$\begin{gathered} -\frac{1}{16} H_{\mu\lambda\sigma} H_{\nu}{}^{\lambda\sigma} g^{ab} \partial_a X_0^{\mu} \partial_b X_0^{\nu} \\ \frac{1}{8} \nabla^{\lambda} H_{\lambda\mu\nu} \epsilon^{ab} \partial_a X_0^{\mu} \partial_b X_0^{\nu} \end{gathered} \tag{5.1.19}$$

Summing up all these contributions gives

$$\begin{aligned}\langle T_{z\bar{z}}(z)\rangle &= \tfrac{1}{4}\left(R_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\sigma}H_{\nu}{}^{\lambda\sigma}\right)\partial_a X_0^\mu \partial^a X_0^\nu + \tfrac{1}{8}\nabla^\lambda H_{\lambda\mu\nu}\epsilon^{ab}\partial_a X_0^\mu \partial_b X_0^\nu \\ &\equiv \tfrac{1}{4}\beta^G_{\mu\nu} g^{ab}\partial_a X_0^\mu \partial_b X_0^\nu + \tfrac{1}{4}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X_0^\mu \partial_b X_0^\nu\end{aligned} \tag{5.1.20}$$

These β coefficients are related to the beta functions associated with the couplings of $G_{\mu\nu}$ and $B_{\mu\nu}$, but they are not exactly the same. For this purpose we will not worry about that. The only relevant feature is that for scale invariance it is the same to guarantee the vanishing of this functions or of the "real" βfunctions.

Until now we considered the worldsheet to be flat and in a flat worldsheet S_3 is zero. By other side, the inclusion of this term in the action will also change the energy-momentum tensor since the variation of the Ricci scalar with respect to the metric is non trivial. It is just a matter of computation to obtain the dilaton contribution to energy-momentum tensor trough variation of the Ricci scalar, to obtain

$$T^{dil}_{ab} = \left(\partial_a \partial_b - g_{ab}\Box\right)\phi(X) \tag{5.1.21}$$

As we can see it has a non null trace

$$T^{dil}_{+-} = \Box\phi(X) \tag{5.1.22}$$

as expected since it is not Weyl invariant already at a classical level. As we stated, this can now be used to cancel the one loop anomalies coming from the classical Weyl invariant terms. This idea is supported by dimensional analysis: for each loop we get a factor of α'. With a T proportional to $\frac{1}{\alpha'}$ this in fact looks like a one loop term (as the ones we calculated coming from true one loop diagrams).

The anomaly is not presented in a covariant form. For this we use the chain rule in the D'Alembertian of the dilaton and then impose the equations of motion for X_0^μ. These equations can be obtained by considering just uniquely the actions S_1 and S_2 since we still work on a plane worldsheet with $S_3 = 0$. They are

$$\Box X_0^\mu = \Gamma^\mu_{\alpha\beta}\partial_a X_0^\alpha \partial^a X_0^\beta - \frac{1}{2}H^\mu{}_{\alpha\beta}\partial_a X_0^\alpha \partial_b X_0^\beta \epsilon^{ab} \tag{5.1.23}$$

Imposing this after the chain rule on T^{dil}_{+-}

$$\left\langle T^{dil}_{+-}\right\rangle = \frac{1}{2}\nabla_\mu\nabla_\nu\phi(X_0)\,\partial_a X_0^\mu \partial^a X_0^\nu - \frac{1}{4}\nabla^\sigma\phi H_{\sigma\mu\nu}\partial_a X_0^\alpha \partial_b X_0^\beta \epsilon^{ab} \tag{5.1.24}$$

The final conditions for Weyl invariance are

$$\begin{aligned}\beta^G_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma} + 2\nabla_\mu\nabla_\nu\phi = 0 \\ \beta^B_{\mu\nu} &= \tfrac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} - \nabla^\sigma\phi H_{\sigma\mu\nu} = 0\end{aligned} \tag{5.1.25}$$

This is the final result regarding a flat worldsheet, but we can only achieve a flat worldsheet by gauge choice if we use Weyl invariance and diffeomorphism invariance. Until we impose the conditions above, Weyl invariance is not guaranteed, so we have no legitimacy to make this assumption. Still we have enough freedom to pick a conformally flat metric $g_{ab} = e^{\Lambda(\sigma)}\eta_{ab}$ to obtain the complete result. Until now it only appeared two coefficients to set to zero: $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$. These will remain unchanged at this order by the introduction of a conformally flat metric, nevertheless it will appear a new coefficient to set up to zero: β^ϕ. This one is missing from the above equations because in flat space the term S_3 of the action is zero, but in a curved space, this term will be multiplied by the worldsheet Ricci scalar to give the anomaly. Thankfully there is no need to make all the calculation again and we will not proceed thought all detail calculation here, but rather present a short description of how to do it. The line of thought is basically the same that was presented until here to obtain equation 5.1.25 plus a couple of subtleties. By direct derivation from the path integral

$$\frac{\delta}{\delta\Lambda(\sigma)} \langle T_{+-}(0)\rangle_{e^{\Lambda(\sigma)}\eta_{ab}} |_{\Lambda=0} = -\frac{1}{4\pi} \langle T_{+-}(\sigma)\, T_{+-}(0)\rangle_{\eta_{ab}} \tag{5.1.26}$$

This way is still possible to keep the calculations using flat space. The way to compute this is to consider again other quantity, in this case $\langle T_{++}(\sigma)\, T_{++}(0)\rangle_{\eta_{ab}}$ and use the conservation of energy-momentum tensor to obtain the previous equation. At the lowest order there is only one diagram contributing (one insertion from $T_{++}(\sigma)$ and other from $T_{++}(0)$). The final result for the two point functions at this order:

$$\langle T_{+-}(\sigma)\, T_{+-}(0)\rangle_{\eta_{ab}} = \frac{\pi D}{12} \Box \delta^{(2)}(\sigma)$$

This is easy to integrate, and writing Λ as a function of the curvature

$$\langle T_{+-}(\sigma)\rangle_{e^{\Lambda(\sigma)}\eta_{ab}} = \frac{D}{24}\sqrt{g}R^{(2)} \tag{5.1.27}$$

There is no field dependence yet, and in fact, at this order, it should not since we did not look to the coupling of the action and used only insertions from the energy-momentum tensor. To go further, we need to calculate two loop diagrams. Some of them will contribute for $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ but at an higher order ($\sim \alpha'$) so they are not considered. These calculations require a lot of effort: subdivergences will appear claiming for renormalization, which will imply the appearance of counterterms to be considered too. One should be careful when treating these divergences to respect the conservation of the energy-momentum tensor. We will not go into more detail here.

The contribution of the dilaton coupling to β^ϕ at this order has two contributions: one is due to the anomaly at the classical level only and may be obtained directly from 5.1.26 calculating the two point functions of T^{dil}_{+-} with itself ($\left\langle T^{dil}_{+-} T^{dil}_{+-}\right\rangle$), the other is a one loop diagram of T^{dil}_{+-} with T_{++}. Providing the conservation equation this contribution gives $\left\langle T^{dil}_{+-} T_{+-}\right\rangle$ (T^{dil}_{+-} with anomalies due to the classical Weyl invariant part). Picking all these contributions

$$\beta^\phi = \frac{D}{6} + \frac{\alpha'}{2}\left(-R + \frac{1}{12}H^{\alpha\beta\gamma}H_{\alpha\beta\gamma} + 4(\nabla\phi)^2 - 4\nabla^2\phi\right) \tag{5.1.28}$$

Even though $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ were not obtained in the same order of expansion on α' when compared with β^ϕ, the physical order is the same because only at order of α' in β^ϕ we consider the derivatives of the spacetime coupling functions as it happens at order α'^0 in $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$.

5.2. Consistency of Weyl invariance conditions

As we discussed in the previous section, to obtain Weyl invariance we should just guarantee that the fields G, B and ϕ satisfy the condition

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\phi = 0 \tag{5.2.1}$$

The natural question is whether there exists such solution or not. At first sight this seems to be impossible: the leading order term in β^ϕ, equation 5.1.28, does not depend on the field, just the dimension. This anomaly appears even with no fields in flat spacetime. Our previous work on Weyl anomaly gives the answer: this constant is eliminated by the anomaly in the ghosts that appear in the path integral quantization. In fact, they are completely decoupled from this degrees of freedom and the only way they can contribute to this $\beta-$functions is through an additive constant. It can be shown that this constant is $-\frac{26}{6}$ and guarantees the Weyl invariance at this order as long as $D = 26$ as we should expect and desire. Thus, for $D = 26$ we forget about that piece.

One way to prove that the Weyl invariance conditions 5.2.1 are consistent with each other is to derive them from an action. A good start to do this is to write the conditions, if possible, in the form of Einstein equations since we might expect to obtain them after the results of the beginning of section 5. This is not hard to do. We can find

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = \frac{1}{4}\left(H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma} - \frac{1}{6}G_{\mu\nu}H^2\right) + 2\nabla^2\phi G_{\mu\nu} - 2\nabla_\mu\nabla_\nu\phi \\ \nabla^\lambda H_{\lambda\mu\nu} = 2\nabla^\lambda\phi H_{\lambda\mu\nu} \\ \nabla^2\phi - 2\left(\nabla\phi\right)^2 = -\frac{1}{2}H^2 \end{cases} \tag{5.2.2}$$

We can define then the energy-momentum tensor for the spacetime:

$$T_{\mu\nu} = \frac{1}{4}\left(H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma} - \frac{1}{6}G_{\mu\nu}H^2\right) + 2\nabla^2\phi G_{\mu\nu} - 2\nabla_\mu\nabla_\nu\phi \tag{5.2.3}$$

For this definition to make sense we should have conservation of this tensor and indeed, using the last two equation from 5.2.2 we can conclude that $\nabla^\mu T_{\mu\nu} = 0$. This shows the consistency of the system of equations that in fact are the equations of motion of the action

$$S = \int d^D X\sqrt{G}e^{-2\phi}\left(R + 4\left(\nabla\phi\right)^2 - \frac{1}{12}H^2\right) \tag{5.2.4}$$

This is very nice: there is an action defined on the D-dimensional spacetime that govern how the fields fluctuate and it is obtained only by consistency conditions for the string. However we can put it in an even better look, in the sense that is more familiar. By a rescaling of the metric

$$G_{\mu\nu} = e^{\frac{4}{2-D}\phi}\bar{G}_{\mu\nu}$$

Knowing how the Ricci scalar changes with a rescaling

$$g_{\mu\nu} \rightarrow e^{2\lambda} g_{\mu\nu} \Rightarrow R \rightarrow e^{2\lambda}\left[R + 2\left(D-1\right)\nabla^2\lambda - \left(D-2\right)\left(D-1\right)\partial_\mu\lambda\partial^\mu\lambda\right]$$

we particularize for this rescaling and find

$$S = \int d^D X \sqrt{\bar{G}} \left(\bar{R} + \frac{4}{D-2}\left(\bar{\nabla}\phi\right)^2 - \frac{1}{12} e^{\frac{8}{D-2}\phi} \bar{H}^2\right) \tag{5.2.5}$$

This has the form of Einstein-Hilbert action, a kinetic term from the dilaton and a “Maxwell term”. We call this action an effective action in the sense that if we derived propagators and interaction vertex from it to calculate scattering amplitudes for G, B and ϕ fields, the result will be the same as if we worked on the operator formalism of string theory for the massless states, at least, at this order. The problem we may face now is that we only computed β functions to the order of α', so, indeed, we cannot guarantee that this is an effective action corresponding to the interaction of massless states, and worst then that, we cannot even guarantee that there exists a solution for all orders to the conditions 5.2.1.

At this stage, one motivation to “go higher order” is to obtain quantum corrections to General Relativity. This will obviously require the non trivial calculation of two loop beta functions. We will just state the result in the vacuum:

$$R_{\mu\nu} + \frac{1}{2}\alpha' R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} = 0 \tag{5.2.6}$$

For further details see[9].

6. Worldsheet two-point functions in AdS_5 from $\mathbb{M}_5$

6.1. Introduction

In this chapter we will work on a particular background within a specified number of dimensions: the five dimensional Anti-de Sitter space (AdS_5). The choice of this geometry relies in the well established result of the exact correspondence of type IIB string theory compactified in $AdS_5 \times S^5$ with four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This is the most famous example of the AdS/CFT correspondence. The nature of this result and the techniques involved have no crucial rule in our goal of calculating the change of dimension of the field operators. As a consequence we just present a brief description about what this correspondence consists and then move to the central topic of this chapter. For a short introduction to AdS space see appendix A.

What is the AdS/CFT correspondence? The AdS/CFT correspondence is a remarkable result first conjectured by Maldacena[5]. It is the establishment of a relation between two types of theories: gauge theories (like Yang-Mills) with gravity theories (like string theory). The example above gave the name to the result but, generally, it should be called gauge/gravity duality since there are examples of such a correspondence in theories where strings are not compactified in an AdS space and the correspondent field theory is not conformally invariant.

Maldacena did not specify, however, which it was the map between the two theories. This was done independently by Witten[18] and Gubser, Klebanov, and Polyakov[19]. These authors established a relation between the fields of the string theory and operators on the boundary of the space.

By other side the partition function of the string is the generating functional for correlation functions of field theory operators. On our previous discussion of string theory we did not specify how to deal with string interactions. It is not clear how one should compute them. The introduction of nonlinear terms in the action is not very plausible since it will always compromise our gauge symmetries that we made such a big effort to keep. The correct answer comes from, besides summing over all the metrics in the path integral, summing also over topologies of the worldsheet, namely, summing over increasingly higher genus (sphere genus 1, torus genus 2,...). We will not specify how to do this, but it will be the partition function constructed in this way that will match the usual generating functional of the field theory.

6.2. Expected values in AdS_5 from Minkowski space

In the final part of appendix A we have obtained the AdS metric as flat metric plus $O\left(\left(\frac{\rho}{R}\right)^2\right)$ in spherical coordinates. We explore now this expansion in 5 dimensions since we shall not need more general cases. We Taylor expand the functions

$$\cosh^2\frac{\rho}{R} = 1 + \left(\frac{\rho}{R}\right)^2 + ...$$

$$\sinh^2\frac{\rho}{R} = \left(\frac{\rho}{R}\right)^2 + \frac{1}{3}\left(\frac{\rho}{R}\right)^4 + ...$$

and obtain

$$ds^2 = -d\tau^2 + d\rho^2 + \rho^2 d\Omega_3^2 - \left(\frac{\rho}{R}\right)^2 d\tau^2 + \frac{R^2}{3}\left(\frac{\rho}{R}\right)^4 d\Omega_3^2$$

We would like to find now the perturbation to the flat metric in Cartesian coordinates instead of spherical ones. We have

$$\rho^2 = \eta_{ij} y^i y^j$$

$$\rho d\rho = \eta_{ij} y^i dy^j$$

$$\eta_{ij} dy^i dy^j = dp^2 + \rho^2 d\Omega_3^2 \Longrightarrow d\Omega_3^2 = \frac{1}{\rho^2}\eta_{ij} dy^i dy^j - \frac{1}{\rho^4}\eta_{ij}\eta_{kl} y^i y^k dy^j dy^l$$

that substituting on ds^2yields

$$ds^2 = -d\tau^2 + \eta_{ij} dy^i dy^j - \frac{1}{R^2}\eta_{ij} y^i y^j d\tau^2 + \frac{1}{3R^2}\left(\rho^2 \eta_{ij} dy^i dy^j - \eta_{ij}\eta_{kl} y^i y^k dy^j dy^l\right)$$

$$= -d\tau^2 + \eta_{ij} dy^i dy^j - \frac{1}{R^2}\eta_{ij} y^i y^j d\tau^2 + \frac{1}{3R^2}\left(\eta_{ij}\eta_{kl} y^k y^l dy^i dy^j - \eta_{ij}\eta_{kl} y^i y^k dy^j dy^l\right)$$

$$= -d\tau^2 + \eta_{ij} dy^i dy^j - \frac{1}{R^2}\eta_{ij} y^i y^j d\tau^2 + \frac{1}{3R^2}\left(\eta_{ij}\eta_{kl} - \eta_{lj}\eta_{ki}\right) y^k y^l dy^i dy^j$$

In these coordinates the elements of the metric are

$$g_{\tau\tau} = -1 - \frac{1}{R^2}\eta_{ij} y^i y^j = -1 - \frac{1}{R^2}\vec{y}^2$$

$$g_{i\tau} = 0$$

$$g_{ij} = \eta_{ij} + \frac{1}{3R^2}\left(\eta_{ij}\eta_{kl} - \eta_{lj}\eta_{ki}\right) y^k y^l = \eta_{ij} + \frac{1}{3R^2}\left(\vec{y}^2 \eta_{ij} - y^i y^j\right)$$

At the origin all the first derivatives vanish. The second derivatives are

$$\begin{aligned} g_{\tau\tau,ij} &= -\tfrac{2}{R^2}\eta_{ij} \\ g_{ij,kl} &= \tfrac{1}{3R^2}\left(2\eta_{kl}\eta_{ij} - \eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}\right) \end{aligned} \tag{6.2.1}$$

These are just properties of the space, we now want to obtain mean values in AdS_5. Consider then some metric in some coordinates taking the form $g_{\mu\nu}(X) \simeq \eta_{\mu\nu} + \frac{1}{2}g_{\mu\nu,\alpha\beta}(X_0)\,X^\alpha X^\beta$ (just like our previous ones). Substituting this in the path integral and denoting the Polyakov action by S_0 one obtains

$$\begin{aligned} \langle \mathfrak{F} \rangle &= \frac{\int DX \mathfrak{F} e^{-S_0 - \frac{1}{4\pi\alpha'} g_{\mu\nu,\alpha\beta}(X_0)\int d^2z \partial X^\mu \bar{\partial} X^\nu X^\alpha X^\beta}}{\int DX e^{-S}} \\ &= \frac{\int DX \mathfrak{F} e^{-S_0}\left(1 - \frac{1}{4\pi\alpha'} g_{\mu\nu,\alpha\beta}(X_0)\int d^2z \partial X^\mu \bar{\partial} X^\nu X^\alpha X^\beta\right)}{\int DX e^{-S}} \\ &\equiv \langle \mathfrak{F} \rangle_0 - \tfrac{1}{4\pi\alpha'} g_{\mu\nu,\alpha\beta}(X_0)\left\langle \mathfrak{F} : \int d^2z \partial X^\mu \bar{\partial} X^\nu X^\alpha X^\beta : \right\rangle_0 \end{aligned} \tag{6.2.2}$$

Here the brackets $\langle\rangle_0$ mean we are calculating the mean value in flat space.

We learn that to calculate 2-point functions in AdS_5 we have just to particularize $\mathfrak{F}$ and the second derivative of the metric. This way, up to second order, we can obtain 2-point functions in AdS_5 by calculating 2 and 3-point functions in flat space.

6.3. Klein-Gordon equation in AdS_5

To calculate the general form of a 2-point function of a vertex operator we cannot consider their form in flat space. Due to the curvature of the space, the form of the vertex operators also changes. To obtain the new operators in AdS_5 one must solve the Klein-Gordon equation. Again the idea is to find a perturbation to Minkowski, so we start with the later one.

Generally one can write the D'Alembert operator as

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) \tag{6.3.1}$$

We consider spherical coordinates in $\mathbb{M}^5$ and insert the ansatz

$$\phi(\tau,\rho,\Omega_3) = e^{-iE\tau} f(\rho)\, Y_l^m(\Omega_3) \tag{6.3.2}$$

where $\nabla^2_{S^3} Y_l^m = -l(l+2)\,Y_l^m$. The radial equation is

$$f''(\rho) + \frac{3}{\rho}f'(\rho) + \left(E^2 - m^2 - \frac{l(l+2)}{\rho^2}\right)f(\rho) = 0 \tag{6.3.3}$$

with general solution

$$f(\rho) = \frac{1}{\rho}\left(AJ_{l+1}\left(\sqrt{E^2-m^2}\rho\right) + BY_{l+1}\left(\sqrt{E^2-m^2}\rho\right)\right)$$

where J and Y are respectively the Bessel functions of the first and second kind. Bessel functions of the second kind are always divergent at the origin so we always take $B=0$. The solution that we will be interested in is

$$f(\rho) = \frac{A}{\rho}J_{l+1}\left(\sqrt{E^2-m^2}\rho\right) \tag{6.3.4}$$

We now want to obtain, perturbatively, the solution for AdS_5. Using the coordinates A.0.12 in 6.3.1 we apply the same ansatz changing only the radial equation to

$$f''(\rho) + \frac{1}{R}\left(\tanh\frac{\rho}{R} + \frac{3}{\tanh\frac{\rho}{R}}\right)f'(\rho) + \left(\left(\frac{E}{\cosh\frac{\rho}{R}}\right)^2 - \frac{l(l+2)}{\left(R\sinh\frac{\rho}{R}\right)^2}\right)f(\rho) = m^2 f(\rho) \tag{6.3.5}$$

Expanding in powers of $\frac{\rho}{R}$

$$f''(\rho) + \left(\frac{3}{\rho} + \frac{2\rho}{R^2}\right)f'(\rho) + \left(E^2 - \frac{l(l+2)}{\rho^2} + \frac{l(l+2)}{3R^2} - \left(E^2 - \frac{l(l+2)}{15R^2}\right)\frac{\rho^2}{R^2}\right)f(\rho) = m^2 f(\rho) \tag{6.3.6}$$

To acquire some intuition about the solution it is useful to write this equation in the Schrödinger form, so we can look to its potential. This is possible through the substitution

$$f(\rho) = \rho^{\alpha}e^{\beta\frac{\rho^2}{R^2}}g(\rho) \tag{6.3.7}$$

with the parameter α and β to be adjusted so there is no term in $g'(\rho)$. After some computation we obtain the values for the parameter ($\alpha = -\frac{3}{2}$, $\beta = -\frac{1}{2}$) and an equation of the form $-g''(\rho) + V(\rho)g(\rho) = m^2 g(\rho)$ where the potential is

$$V(\rho) = -\left(l(l+2) + \frac{3}{4}\right)\rho^{-2} + \left(E^2 + \frac{l(l+2)}{3R^2} - \frac{5}{R^2}\right) - \left(\frac{1}{R^2}\left(E^2 + \frac{l(l+2)}{15R^2}\right) + \frac{1}{R^4}\right)\rho^2 \tag{6.3.8}$$

Some plots of this potential for different parameters are presented in figure 6.3.1. From here it is worth to note by one side the locality of the solutions (in fact, *AdS* is a gravitational box) and by other the fact that *AdS* gets closer to flat space as we are increasing the radius (as we already knew). We also note that the potential is always divergent at the origin.

Now we will seek an explicit perturbed solution for the Klein-Gordon equation. To get this result it is useful to work in global coordinates A.0.15. By other side, because we learned to calculate the 2-point functions in our previous coordinates in section 6.2, we will get back to them by the relation among the radial variable:

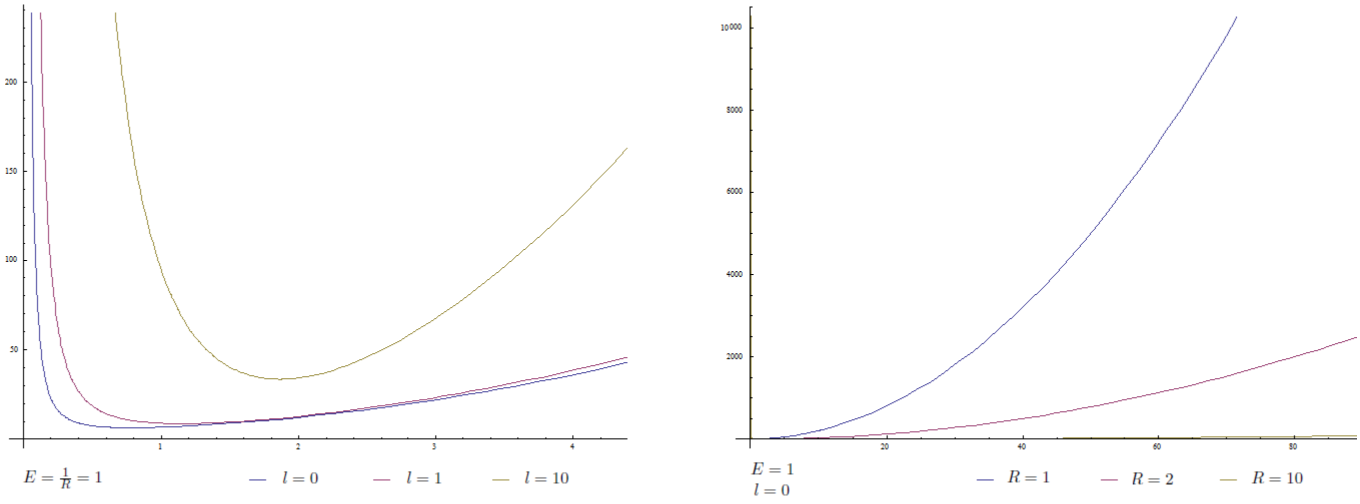


Figure 6.3.1.: Plots of the potential 6.3.8 varying the angular momentum on the left and the radius of AdS on the right.

$$\mu = R \arccos\left(\frac{1}{\cosh\frac{\rho}{R}}\right) \tag{6.3.9}$$

In these coordinates we may use the same ansatz 6.3.2. The radial wave equation reads

$$\frac{R^2}{(\tan\frac{\mu}{R})^{d-1}}\partial_\mu\left(\left(\tan\frac{\mu}{R}\right)^{d-1}\partial_\mu\right)\chi + \left(E^2R^2 - l\,(l+2)\csc^2\frac{\mu}{R} - m^2R^2\sec^2\frac{\mu}{R}\right)\chi = 0 \tag{6.3.10}$$

χ being the radial function. Substituting $\chi(\mu) = \left(\cos\frac{\mu}{R}\right)^{2h}\left(\sin\frac{\mu}{R}\right)^{2b} q(\mu)$ and with $y = \sin^2\frac{\mu}{R}$ the equation takes a simpler form

$$y\,(y-1)\,\partial_y^2 q + (2b + 2 - (2h + 2b + 1)\,y)\,\partial_y q - \left((h+b)^2 - \frac{E^2R^2}{4}\right)q = 0 \tag{6.3.11}$$

This is an hypergeometric equation. We imposed conditions on h and b, namely $h\,(h-2) = \frac{m^2R^2}{4}$ and $2b\,(2b+2) = l\,(l+2)$. The solution for these parameters is

$$h_\pm = 1 \pm \frac{\sqrt{4 + m^2R^2}}{2}, \; b = \frac{l}{2}, \; -\frac{1}{2}\,(l+2) \tag{6.3.12}$$

There are two independent solutions corresponding to the two possibilities for b. Because of the two solutions for h, it seems we have four independent solutions. It is not true, two of them are equal to the other two, which means we can pick without loss of generality h_+. We can see the independent solutions in [1] (equations 15.5.4 and 15.5.9), but we are only interested in the case of regularity at the origin that is:

$$\chi(\mu) = \left(\cos\tfrac{\mu}{R}\right)^{2+\sqrt{4+(Rm)^2}}\left(\sin\tfrac{\mu}{R}\right)^{l}$$
$$\times{}_2F_1\left(1+\tfrac{1}{2}\sqrt{4+(Rm)^2}+\tfrac{1}{2}(l+ER),1+\tfrac{1}{2}\sqrt{4+(Rm)^2}+\tfrac{1}{2}(l-ER),l+2;\sin^2\tfrac{\mu}{R}\right) \tag{6.3.13}$$

where ${}_2F_1$ is the hypergeometric function. For further detail on the solutions and their stability see [6].

Expanding the solutions

We will now compare the series expansions on the two different spaces. We will only present the case in which $l=0$. For the Minkowski case, we expand equation 6.3.4 setting $A=\frac{2}{\sqrt{E^2-m^2}}$

$$1+\frac{1}{8}\left(E^2-m^2\right)\rho^2+\frac{1}{192}\left(-E^2+m^2\right)^2\rho^4+\mathcal{O}\left(\rho^6\right) \tag{6.3.14}$$

In AdS_5 we put 6.3.9 into 6.3.13 and expand. The result is

$$1-\frac{1}{8}\left(E^2-m^2\right)\rho^2+\left(\frac{1}{192}\left(E^2-m^2\right)^2+\frac{3E^2-m^2}{48R^2}\right)\rho^4+\mathcal{O}\left(\rho^6\right) \tag{6.3.15}$$

For $E\neq m$

$$1+\frac{1}{8}\left(-E^2+m^2\right)\rho^2+\frac{1}{192}\left(-E^2+m^2\right)^2\left[1+\frac{12E^2-4m^2}{R^2\left(E^2-m^2\right)^2}\right]\rho^4+\mathcal{O}\left(\rho^6\right) \tag{6.3.16}$$

When comparing to flat space the first correction appears in the fourth order of the ρ expansion. Joining this result with the one from section 6.2 we can calculate generally the first orders of the 2-point function of any vertex operator.

From equation 6.2.2 we see that the first correction is, in principle, of order $\frac{1}{R^2}$ coming from the contraction of the two terms of ρ^4 in $\langle\mathfrak{F}\rangle_0$ and from the contraction of the ρ^2 and ρ^4 terms in $\left\langle\mathfrak{F}:\int d^2z\partial X^\mu\bar{\partial}X^\nu X^\alpha X^\beta:\right\rangle_0$.

As an example of how we can obtain these corrections we calculate explicitly the 2-point function of the Lagrangian.

6.4. Anomalous dimension of the Lagrangian operator

Here we drop the pre-factor in the Lagrangian in order to avoid heavy notation. We consider then $\partial X^\mu(z,\bar{z})\,\bar{\partial}X^\nu(z,\bar{z})\,g_{\mu\nu}$. From table 3.1 we know that in the free theory[1] the two point function

$$\left\langle:\partial X^\mu(z,\bar{z})\,\bar{\partial}X^\nu(z,\bar{z})\,\eta_{\mu\nu}::\partial X^\sigma(0,0)\,\bar{\partial}X^\beta(0,0)\,\eta_{\sigma\beta}:\right\rangle_0$$

[1] We are working on a curved space, namely AdS_5. When we refer to results in the "free theory" we are always referring to $\mathbb{M}_5$.

must go as $\frac{1}{|z|^4}$. From Wick contractions we can obtain the pre factor:

$$\left\langle : \partial X^{\mu}\left(z,\bar{z}\right)\bar{\partial}X^{\nu}\left(z,\bar{z}\right)\eta_{\mu\nu} :: \partial X^{\sigma}\left(0,0\right)\bar{\partial}X^{\beta}\left(0,0\right)\eta_{\sigma\beta} : \right\rangle_0 = 5\left(\frac{\alpha'}{2}\right)^2\frac{1}{|z|^4} \tag{6.4.1}$$

Because AdS is flat up to second order on the curvature, it is expected to obtain a correction to the power of $|z|$ presented in the previous expression. In what follows we will describe a detailed derivation of this result.

As we saw we have to consider two corrections: by one side we have the change in the Lagrangian that is no longer the same operator ($\partial X^{\mu}\left(z,\bar{z}\right)\bar{\partial}X^{\nu}\left(z,\bar{z}\right)\eta_{\mu\nu} \to \partial X^{\mu}\left(z,\bar{z}\right)\bar{\partial}X^{\nu}\left(z,\bar{z}\right)g_{\mu\nu}$), by other we have the change in the way we calculate expected values. The change on the Lagrangian does not contribute at this order ($\frac{1}{R^2}$). Schematically we would have terms of the form $\left\langle : \partial X^{\mu}\bar{\partial}XXX :: \partial X\bar{\partial}X : \right\rangle_0$ which are always zero[2].

So we are considering only equation 6.2.2 with the Lagrangian in flat space. We already have the first term, lets move on to the other.

$$\left\langle : \partial X^{\mu}\left(z,\bar{z}\right)\bar{\partial}X^{\nu}\left(z,\bar{z}\right)\eta_{\mu\nu} :: \partial X^{\sigma}\left(0,0\right)\bar{\partial}X^{\beta}\left(0,0\right)\eta_{\alpha\beta} :: \partial X^{\alpha_1}\left(z',\bar{z}'\right)\bar{\partial}X^{\alpha_2}\left(z',\bar{z}'\right)X^{\alpha_3}\left(z',\bar{z}'\right)X^{\alpha_4}\left(z',\bar{z}'\right) : \right\rangle_0$$

$$\begin{aligned} &= \eta_{\mu\nu}\eta_{\sigma\beta}\left(-\tfrac{\alpha'}{2}\right)^4\Big[\left(\eta^{\mu\alpha_1}\eta^{\nu\alpha_2}\eta^{\sigma\alpha_3}\eta^{\beta\alpha_4} + \text{Switch }\alpha_3\leftrightarrow\alpha_4\right)\tfrac{1}{|z-z'|^4}\tfrac{1}{|z'|^2} \\ &+ \left(\eta^{\mu\alpha_1}\eta^{\nu\alpha_3}\eta^{\sigma\alpha_4}\eta^{\beta\alpha_2} + \text{Switch }\alpha_3\leftrightarrow\alpha_4\right)\tfrac{1}{(z-z')^2(\bar{z}')^2}\tfrac{1}{(\bar{z}-\bar{z}')z'} \\ &+ \left(\eta^{\mu\alpha_3}\eta^{\nu\alpha_4}\eta^{\sigma\alpha_1}\eta^{\beta\alpha_2} + \text{Switch }\alpha_3\leftrightarrow\alpha_4\right)\tfrac{1}{|z'|^4}\tfrac{1}{|z-z'|^2} \\ &+ \left(\eta^{\mu\alpha_3}\eta^{\nu\alpha_2}\eta^{\sigma\alpha_1}\eta^{\beta\alpha_4} + \text{Switch }\alpha_3\leftrightarrow\alpha_4\right)\tfrac{1}{(z')^2(\bar{z}-\bar{z}')^2}\tfrac{1}{(z-z')}\tfrac{1}{\bar{z}'} \end{aligned} \tag{6.4.2}$$

$$\begin{aligned} &= \left(-\tfrac{\alpha'}{2}\right)^4\Big[2\eta^{\alpha_1\alpha_2}\eta^{\alpha_3\alpha_4}\left(\tfrac{1}{|z-z'|^4}\tfrac{1}{|z'|^2} + \tfrac{1}{|z'|^4}\tfrac{1}{|z-z'|^2}\right) \\ &+ \left(\eta^{\alpha_1\alpha_3}\eta^{\alpha_2\alpha_4} + \eta^{\alpha_1\alpha_4}\eta^{\alpha_2\alpha_3}\right)\left(\tfrac{1}{(z-z')^2(\bar{z}')^2}\tfrac{1}{(\bar{z}-\bar{z}')z'} + \tfrac{1}{(z')^2(\bar{z}-\bar{z}')^2}\tfrac{1}{(z-z')}\tfrac{1}{\bar{z}'}\right)\Big] \end{aligned}$$

In the first step we simply exhausted all the possible ways of contracting all the fields with each other. The second step just exhibits the sum over the indices $\mu,\ \nu,\ \alpha,\ \beta$. The sums on the α indices are

$$\begin{aligned} \eta^{\alpha_1\alpha_2}\eta^{\alpha_3\alpha_4}g_{\alpha_1\alpha_2,\alpha_3\alpha_4} &= 0 \\ \eta^{\alpha_1\alpha_3}\eta^{\alpha_2\alpha_4}g_{\alpha_1\alpha_2,\alpha_3\alpha_4} &= -\tfrac{4}{R^2} \end{aligned} \tag{6.4.3}$$

which makes the first term zero. In the second the pre-factor $\eta^{\alpha_1\alpha_3}\eta^{\alpha_2\alpha_4} + \eta^{\alpha_1\alpha_4}\eta^{\alpha_2\alpha_3}$ get the same result because $g_{\alpha_1\alpha_2,\alpha_3\alpha_4}$ is symmetric under $\alpha_3\leftrightarrow\alpha_4$. This leads to

[2] In an explicit way, we must contract fields on the right with field on the left. At the best we will always get the mean value of a normal product among two fields which is always zero.

$$\begin{aligned}\left\langle : \partial X^{\mu}(z,\bar{z})\,\bar{\partial} X^{\nu}(z,\bar{z})\,\eta_{\mu\nu} :: \partial X^{\sigma}(0,0)\,\bar{\partial} X^{\beta}(0,0)\,\eta_{\sigma\beta} : \right\rangle &= 5\left(\tfrac{\alpha'}{2}\right)^{2}\tfrac{1}{|z|^{4}} \\ +\tfrac{\alpha'^{3}}{4\pi R^{2}}\int d^{2}z'\left(\tfrac{1}{(z-z')^{2}(\bar{z}')^{2}}\tfrac{1}{(\bar{z}-\bar{z}')z'}+\tfrac{1}{(z')^{2}(\bar{z}-\bar{z}')^{2}}\tfrac{1}{(z-z')\bar{z}'}\right)&\end{aligned} \tag{6.4.4}$$

We can write the argument of the integral as

$$\frac{(\bar{z}-\bar{z}')\,z'+(z-z')\,\bar{z}'}{|z-z'|^{4}\,|z'|^{4}}=\frac{\bar{z}z'+z\bar{z}'-2\,|z'|^{2}}{|z-z'|^{4}\,|z'|^{4}}=\frac{|z|^{2}-|z-z'|^{2}-|z'|^{2}}{|z-z'|^{4}\,|z'|^{4}}$$

where we have used that $\bar{z}z'+z\bar{z}'=|z|^{2}+|z'|^{2}-|z-z'|^{2}$. To perform the integral we consider $z'\sim 0$ where $\frac{|z|^{2}-|z-z'|^{2}-|z'|^{2}}{|z-z'|^{4}|z'|^{4}}\simeq-\frac{1}{|z|^{4}|z'|^{2}}$ so the result will be a logarithm

$$\left\langle : \partial X^{\mu}(z,\bar{z})\,\bar{\partial} X^{\nu}(z,\bar{z})\,\eta_{\mu\nu} :: \partial X^{\sigma}(0,0)\,\bar{\partial} X^{\beta}(0,0)\,\eta_{\sigma\beta} : \right\rangle = 5\left(\frac{\alpha'}{2}\right)^{2}\frac{1}{|z|^{4}}\left(1-\frac{4\alpha'}{5R^{2}}\log|\Lambda z|\right) \tag{6.4.5}$$

This allows us to get a prediction of the anomalous dimension of this operator. Namely, with $\left(1-\frac{4\alpha'}{5R^{2}}\log|\lambda z|\right)\sim e^{-\frac{4\alpha'}{5R^{2}}\log|\Lambda z|}$ we find

$$\left\langle : \partial X^{\mu}(z,\bar{z})\,\bar{\partial} X^{\nu}(z,\bar{z})\,\eta_{\mu\nu} :: \partial X^{\sigma}(0,0)\,\bar{\partial} X^{\beta}(0,0)\,\eta_{\sigma\beta} : \right\rangle = \frac{5}{\Lambda^{\frac{4\alpha'}{5R^{2}}}}\left(\frac{\alpha'}{2}\right)^{2}\frac{1}{|z|^{4+\frac{4\alpha'}{5R^{2}}}} \tag{6.4.6}$$

We find the effect of curvature on the weight of the operator at this order $4\to 4+\frac{4\alpha'}{5R^{2}}$. We naturally recover the flat space result for $R^{2}\gg\alpha'$.

7. Conclusion

We have seen how one can construct a quantum theory of strings from a classical action. We followed three different procedures that deal in a different way with ghosts, but in the end are equivalent. In light-cone quantization the critical dimension is obtained by imposing the Lorentz invariance of the theory that is not clearly explicit. In the case of the path integral quantization we must require that the ghost Weyl anomaly cancels the anomaly coming from the physical degrees of freedom. In both cases the value is the same: $D = 26$. When constructing the spectrum of the theory we found a tachyon state and three massless states with particular emphasis for the graviton.

We have also seen some important properties of conformal field theories. Particularly there is a map between state and operators. This map was taken explicitly when working with the Polyakov action, which allowed us to identify, in chapter 5, the introduction of a curved metric in the Polyakov action with the introduction of a coherent state of gravitons. The Weyl invariance is broken for general metrics, but we see that we guarantee it by requiring Einstein equations to hold. We followed the same procedure with the other massless fields and proved the consistency of the equations resulting from Weyl invariance requirements. We also stated the first (quantum) correction to Einstein equations.

The dimension of the vertex operators, dual to the states of the strings, are well known in flat space but not in AdS_5.We described an explicit procedure to calculate 2-point functions in AdS when working in a particular set of coordinates. We then solved the Klein Gordon equation perturbatively. As an explicit example we obtained the anomalous dimension of the Lagrangian operator, which gives in this way the correction to flat space.

A. The AdS space

The Anti-de-Sitter space is well known in General Relativity as opposition to the de-Sitter space. While this last one is a solution of the Einstein equation in vacuum with positive cosmological constant, the Anti-de Sitter space is the solution for negative constant.

In this framework it is more useful to consider AdS space by embedding and we will not worry about the proof of the equivalence of this statement (see for example [14]). We define it as the $d+1$ submanifold embedded in the $d+2$ manifold that satisfies the condition

$$X^\alpha X^\beta \eta_{\alpha\beta} = -R^2 \tag{A.0.1}$$

R is the radius of AdS and the metric $\eta_{\alpha\beta}$ is given by

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \tag{A.0.2}$$

To find the isometries group of this space we search for the transformation $X^\alpha \to X'^\alpha = \Lambda^\alpha{}_\beta X^\beta$ that leaves this metric unchanged, that, by construction, is precisely $SO\,(2,d)$ (excluding the cases of inversion of coordinates in which the determinant of the transformation is -1). We will start by solving the equation by a particular choice: Poincaré coordinates. For that let us use light-cone coordinates on the $d+2$ manifold.

$$\begin{aligned} u &= \tfrac{X^0-X^d}{R} & \qquad X^0 &= \tfrac{v+u}{2}R \\ v &= \tfrac{X^0+X^d}{R} & \qquad X^d &= \tfrac{v-u}{2}R \\ x^i &= \tfrac{X^i}{u} \qquad \Longleftrightarrow & \qquad X^i &= ux^i \\ t &= \tfrac{X^{d+1}}{u} & \qquad X^{d+1} &= ut \end{aligned} \tag{A.0.3}$$

The hyperboloid equation is now $-\left(\frac{v+u}{2}\right)^2 - \left(\frac{tu}{R}\right)^2 + \left(\frac{v-u}{2}\right)^2 + \frac{u^2\vec{x}^2}{R^2} = -1$. Rearranging:

$$v = \frac{1}{u} + \left(\frac{\vec{x}^2 - t^2}{R^2}\right) u \tag{A.0.4}$$

The transformation matrix is given by

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \frac{R}{2} & \frac{R}{2} & & \\ -\frac{R}{2} & \frac{R}{2} & & \\ x^i & & uI_{d-1} & \\ t & & & u \end{pmatrix} \tag{A.0.5}$$

Leading to the metric on these coordinates:

$$g_{\mu\nu} = \begin{pmatrix} x^2 - t^2 & -\frac{R^2}{2} & ux_i & -ut \\ -\frac{R^2}{2} & 0 & & \\ ux_i & & u^2 I_{d-1} & \\ -ut & & & -u^2 \end{pmatrix} \tag{A.0.6}$$

This metric is used to write the element ds^2. With equation A.0.4 it reads

$$ds^2 = \frac{R^2}{u^2} du^2 + u^2 \left(-dt^2 + d\vec{x}^2\right) \tag{A.0.7}$$

The Poincaré coordinates are obtained making the transformation $u = \frac{R}{r}$ giving

$$ds^2 = \frac{R}{r^2} \left(-dt^2 + dr^2 + d\vec{x}^2\right) \tag{A.0.8}$$

With $r \in \,]0, +\infty[$ we conclude that the metric is conformally flat. This assumes that $X^0 - X^d > 0$ and gives only one half of the hyperboloid. We can equally consider the symmetric case to get the other half, which means we need at least two Poincaré charts to cover all *AdS* space. The singularity at $r = 0$ is just a coordinate singularity.

Another possibility is to make the substitution

$$\begin{aligned} X^0 &= R\cosh\left(\tfrac{\rho}{R}\right)\cos\left(\tfrac{\tau}{R}\right) \\ X^i &= R\sinh\left(\tfrac{\rho}{R}\right)\Omega^i \\ X^{d+1} &= R\cosh\left(\tfrac{\rho}{R}\right)\sin\left(\tfrac{\tau}{R}\right) \end{aligned} \tag{A.0.9}$$

solving the hyperboloid equation A.0.1 in a more intuitive way, where $i = 1, ..., d$ and the new variables domains are:

$$\rho \geq 0,\ \ 0 \leq \tau \leq 2\pi R,\ \ -1 \leq \Omega^i \leq 1 \tag{A.0.10}$$

Substituting this on the hyperboloid condition leads to $-\cosh^2 \frac{\mu}{R} + \Omega_i \Omega^i \sinh^2 \frac{\mu}{R} = -1$ which implies

$$\Omega_i \Omega^i = 1 \tag{A.0.11}$$

Now the element ds^2 in *AdS* and in the new coordinates is obtained as before (using the relation $\Omega_i d\Omega^i = 0$).

$$ds^2 = -\cosh^2 \frac{\rho}{R} d\tau^2 + d\rho^2 + R^2 \sinh^2 \frac{\rho}{R} d\Omega_{d-1}^2 \tag{A.0.12}$$

For $\rho \ll R$ this metric is schematically of the form

$$ds^2 = -d\tau^2 + d\rho^2 + \rho^2 d\Omega_{d-1}^2 + O\left(\left(\frac{\rho}{R}\right)^2\right) \tag{A.0.13}$$

Identifying the τ coordinate with time generates a causality problem (because of the domain of τ we have time-like closed curves). To avoid this we set $0 < \tau < +\infty$.

The third and last coordinate system presented here has the advantage of covering all space with just one chart. They are defined as

$$\begin{aligned} X^0 &= R \sec \tfrac{\mu}{R} \cos \tfrac{\tau}{R} \\ X^i &= R \tan \tfrac{\mu}{R} \Omega^i \\ X^{d+1} &= R \sec \tfrac{\mu}{R} \sin \tfrac{\tau}{R} \end{aligned} \tag{A.0.14}$$

The coordinates μ, τ and Ω^i are called global coordinates. The metric reads

$$ds^2 = \frac{1}{\cos^2 \mu} \left(-d\tau^2 + d\mu^2 + \sin^2 \mu d\Omega_{d-1}^2\right) \tag{A.0.15}$$

In these coordinates the spacial infinity, the boundary of AdS, is identified with the hypersurface $\mu = \frac{\pi}{2}$.

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